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Application of the normal wave method to calculations of VLF electromagnetic fields in the anisotropic two-dimensionally inhomogeneous Earth-ionosphere waveguide

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[1] A systematization of the available methods and algorithms used for calculations of amplitude and phase of electromagnetic fields of VLF range in the waveguide channel Earth-ionosphere is given. The presented method is applicable to a wide class of profiles of the electron concentration and collision frequency in the ionospheric D region and takes into account both the dependence of the waveguide channel characteristics along the path and the medium anisotropy due to the influence of the geomagnetic field. The use of the effective height to improve the convergence of the iteration process of computation of eigenvalues and of the bivector for determination of the reflective parameters of the anisotropic ionosphere are the characteristic features of the proposed method. The above indicated features provide a base for realization of the numerical algorithm. *INDEX TERMS:* 6964 Radio Science: Radio wave propagation; 2403 Ionosphere: Active experiments; 2471 Ionosphere: Plasma waves and instabilities; *KEYWORDS:* Earth–ionosphere waveguide; ELF emission; normal waves method.

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1. Introduction

[2] The paper describes the mathematical foundation of the effective algorithm of calculations of components of electric and magnetic field intensity in the 5–30 kHz range generated by a short antenna located at some altitude $d \leq 50$ km over the Earth surface. The distance to the observational point $R \gg d$ and $R \gg \lambda$ (where λ is the wavelength of the electromagnetic field in the vacuum). The main mechanism of VLF propagation [*Budden*, 1961a, 1961b; *Krasnushkin and Yablochkin*, 1963; *Makarov et al.*, 1993; *Wait*, 1962], at which the field is presented as an expansion in terms of modes (eigenfunctions of the discrete spectrum of the lateral operator) propagating at the shortest way between the corresponding points, is taken into account.

[3] The receiving antenna is short and may be located at some height $d_a \leq 50$ km or some depth $d_a < 0$, assuming that $|d_a| \ll R$. The algorithm is based on the path approximation at which possible changes in the waveguide properties along the geodesic line are taken into account, but their possible changes in the lateral direction being neglected.

2. Description of the Waveguide Channel Model

[4] The Earth surface is modeled by a sphere of the radius a, at which the impedance conditions are taken. In this case we exclude out of the regime the waves propagating mainly under the Earth surface because they are insignificant at distances R exceeding the thickness of the skin layer in the Earth. For determination of the impedance in any observational point, a model of the Earth homogeneous by the depth described by Morgan [*Morris and Lha*, 1974] is used.

[5] Over the Earth surface, we have an ionized medium located within a constant magnetic field. The Earth magnetic field is presented in the form $\mathbf{B}_E = \nabla u$, where the potential u is approximated by the expression

$$u(r, \Theta_g, \varphi_g) = \sum_{n=1}^{6} \frac{a^{n+2}}{r^{n+1}} \sum_{m=0}^{n} (a_{nm} \cos m\varphi_g + b_{nm} \sin m\varphi_g) P_{nm}(\sin \Theta_g)$$

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where Θ_g and φ_g are the geographic latitude and longitude, respectively, a is the Earth radius, $P_{n0}(\sin\Theta_g) = P_n(\sin\Theta_g)$ is the Legendre polynomial, and $P_{nm}(\sin\Theta_g)$ is the associated Legendre function. The coefficients a_{nm} and b_{nm} different from zero are presented below expressed in nT:

$$\begin{array}{lll} a_{10} = -30,339 & a_{20} = -1654 & a_{30} = +1297 \\ a_{11} = -2123 & a_{21} = +1222 & a_{31} = -588 \\ & a_{22} = +452 & a_{32} = +118 \\ & & a_{33} = +44 \end{array}$$

$$\begin{array}{lll} a_{40} = +958 & a_{50} = -223 & a_{60} = +47 \\ a_{41} = +180 & a_{51} = +65 & a_{61} = +9 \\ a_{42} = +26 & a_{52} = +8 & \\ a_{43} = -6 & & a_{63} = -1 \\ a_{44} = +2 & & \end{array}$$

$$\begin{array}{lll} b_{11} = +5758 & b_{21} = -819 & b_{31} = -116 \\ & b_{22} = +38 & b_{32} = +22 \\ & & b_{33} = -9 \end{array}$$

$$\begin{array}{lll} b_{41} = +33 & b_{51} = +3 & b_{61} = -2 \\ b_{42} = -15 & b_{52} = +4 & b_{62} = +3 \\ & b_{53} = -1 & \\ b_{44} = -2 & & \end{array}$$

The concentration and collision frequency of electrons are functions of the height over the Earth surface and are taken according to any known model.

[6] In undisturbed conditions, the plasma frequency ω_{pl} at altitudes up to 50 km over the Earth surface is considerably less than the electron collision frequency with neutral particles, so in the near-Earth region the relative dielectric permittivity of the medium ε does not differ from its value in the vacuum. So in the used model the lower boundary of the ionosphere is determined as the level where $|\varepsilon(r_l) - 1| < p1$. In the terrestrial ionosphere there is always such a level r_u, above which the characteristic scale of changes in the medium properties l ($l = \max|(1/\varepsilon_{ik})(\partial\varepsilon_{ik}/\partial r)|^{-1}$, where ε_{ik} is the tensor of relative dielectric permittivity of the ionosphere) is much larger than the characteristic scale of the field changes in r, so calculating this scale $1/h$ one may take the medium as homogeneous (Booker roots) [*Budden*, 1961a]

$$\frac{h}{k_0} \cong \Big|\frac{\omega_{pl}^2}{\omega^2|\mathbf{Y}|}\Big|^{1/2} \qquad \text{where } \mathbf{Y} = \frac{|q_e|}{m_e\omega}\mathbf{B}_E$$

k_0 is the wave number in the vacuum, $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$, ε_0 is the dielectric permittivity of the vacuum, μ_0 is the magnetic permittivity of the vacuum, q_e is the electron charge value, ω is the circular frequency of the electromagnetic field, m_e is the electron mass, $\mathbf{B}_E$ is the magnetic induction of the Earth field, and at $r \geq r_u$ the inequality $h/k_0 \gg 1$ should be fulfilled. Then above r_u the field structure may be taken as a plain wave propagating vertically independently of the field structure at $r < r_u$ and independently of the source of excitation of the electromagnetic field.

[7] We chose the level r_u in such a way that $h/k_0 = p2$ were large enough. Actually, the accuracy of calculation of eigenvalues of the normal waves is determined by the choice of r_l and r_u (the boundaries of the ionospheric layer called the region important for propagation). In particular, the choice of the $p1 = 10^{-4}$ and $p2 = 30$ parameters provides the relative error of calculation of the eigennumber not more than 10^{-4}.

[8] The problem of calculation of the fields within the waveguide channel Earth-ionosphere excited by the antenna located at some height over the Earth surface is formulated. The antenna orientation is given by the angle Θ_p relative to the vertical axis and by the azimuthal angle φ_p, counted clockwise from the direction to the receiver. Jl_p is the current moment of the transmitting antenna. In the observational point, the field is received by the antenna located at any level relative to the Earth surface. The orientation of the receiving antenna is given by the angle Θ_a relative the vertical and by the azimuth angle φ_a, counted clockwise from the transmitter-receiver direction. The induced electromotive force at the short receiving antenna is calculated by the formula

$$U_a = l_a(\tilde{\mathcal{E}}_r \cos\Theta_a + \tilde{\mathcal{E}}_\Theta \sin\Theta_a \cos\varphi_a + \tilde{\mathcal{E}}_\varphi \sin\Theta_a \sin\varphi_a)$$

where l_a is the virtual height of the short receiving antenna, and $\tilde{\mathcal{E}}_r$, $\tilde{\mathcal{E}}_\Theta$, and $\tilde{\mathcal{E}}_\varphi$ are the corresponding components of the electric field intensity in the place of the receiving antenna location.

[9] Mathematically we formulate the problem for a vertical dipole and after its solution we will form the field for an arbitrary oriented short antenna applying the generalized reciprocity theorem.

3. Mathematical Formulation of the Problem

[10] We chose the spherical coordinate system Θ, φ, r, the $\Theta = 0$ axis passing through the vertical dipole. The source excites the falling field independent of φ, so in the chosen model of a one-dimensional irregular waveguide we have an axis-symmetric problem. We write the Maxwell equation in the matrix form [*Felsen and Marcuvitz*, 1973; *Lutchenko and Bulakh*, 1986], taking the following dependence on time $\exp(-i\omega t)$:

$$\tilde{K}\mathbf{\Phi} = -i\frac{\partial}{\partial\Theta}\Gamma\mathbf{\Phi} - i\mathbf{J} \tag{1}$$

where

$$\tilde{\mathbf{\Phi}} = \begin{pmatrix} \tilde{\mathcal{E}} \\ \tilde{\mathcal{H}} \end{pmatrix}$$

$$\tilde{\mathbf{\Phi}} = \frac{1}{\sqrt{\sin\Theta}}\mathbf{\Phi}$$

$$\mathbf{\Phi} = \begin{pmatrix} \mathcal{E} \\ \mathcal{H} \end{pmatrix}$$

$$\mathbf{\Phi} = (\mathcal{E}_\Theta, \mathcal{E}_\varphi, \mathcal{E}_r, \mathcal{H}_\Theta, \mathcal{H}_\varphi, \mathcal{H}_r)^T$$

$\tilde{\mathcal{E}}$ is the electric field intensity vector (V m^{-1}), $\tilde{\mathcal{H}}$ is the magnetic field intensity (A m^{-1}), $\mathcal{H} = \sqrt{(\mu_0/\varepsilon_0)}\tilde{\mathcal{H}}\sqrt{\sin\Theta}$, μ_0 and ε_0 are the magnetic and dielectric constants of the vacuum, respectively, $\mathcal{E} = \tilde{\mathcal{E}}\sqrt{\sin\Theta}$,

$$\tilde{K} = \begin{pmatrix} k_0 r\hat{\varepsilon} & -i\tilde{\nabla}_t \\ i\tilde{\nabla}_t & k_0 r I \end{pmatrix}$$

$$\tilde{\nabla}_t = \begin{pmatrix} 0 & -\dfrac{\partial}{\partial r}r & 0 \\ \dfrac{\partial}{\partial r}r & 0 & \dfrac{1}{2}\cot\Theta \\ 0 & \dfrac{1}{2}\cot\Theta & 0 \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix} \qquad (2)$$

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{J} = r\sqrt{\sin\Theta}\begin{pmatrix} \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\mathbf{j}^e \\ \mathbf{j}^m \end{pmatrix}$$

I is a unit matrix, k_0 is the wave number in the vacuum $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$, $\hat{\varepsilon}$ is the dimensionless tensor of the dielectric permittivity of the magnetoactive medium depending on r and Θ, $\mathbf{j}^m$ is the density of the external magnetic current, $\mathbf{j}^e$ is the density of the external electric current (A m^{-2}) and in the case of a vertical point-like electric dipole located over the Earth surface at $r = b$

$$\mathbf{j}^e = \frac{Jl_p}{2\pi r^2 \sin\Theta}\delta(r-b)\delta(\Theta-\varepsilon)\mathbf{l}_r$$

J is the current at the antenna input, l_p is the antenna virtual height, and $\mathbf{l}_r$ is a unit vector. The problem solution $\tilde{\mathbf{\Phi}}$ has to satisfy the impedance boundary conditions at $r = a$, the conditions of a field decrease at Im$k_0 > 0$ and $r \to \infty$, and its boundedness at $\Theta = 0$ and $\Theta = \pi$. In the accepted model Im$k_0 = 0$, however for choosing the solution, the commonly accepted [*Makarov et al.*, 1993] principle of the limiting amplitude in which a presence of losses in the medium leading to Im$k_0 > 0$ is used. After construction of unambiguous solution, we come back to the model Im$k_0 = 0$ and $\varepsilon = 0$.

[11] The solution of system (1) is constructed by the cross-section method [*Katsenelenbaum*, 1961], presenting the solution in the form of the expansion in terms of orthogonal system of functions $\mathbf{\Psi}_m$

$$\mathbf{\Phi}(\mathbf{r}, \Theta) = \sum_{m>0} A_m(\Theta)\mathbf{\Psi}_m(r, \Theta) \qquad (3)$$

As $\mathbf{\Psi}_m$, the eigenfunctions of the lateral operator $\tilde{K}$ for the regular waveguides of comparison are chosen. The waveguides of comparison are spherical waveguides with the polar axis coinciding with the axis of the initial waveguide and the dielectric permittivity $\hat{\varepsilon}(r)$ which depends only on the coordinate r and coincides to the dielectric permittivity of the initial waveguide in the Θ cross section. To construct the eigenfunctions in the regular waveguides of comparison, we consider homogeneous Maxwell equations and take approximately that $(d\mathbf{A}_m/d\Theta) = i\nu_m\mathbf{A}_m$, where m is the number of the mode. Then we neglect $\cot\Theta$ in the operator $\tilde{\nabla}_t$ assuming that $|\cot\Theta| \ll |\nu_m\ |, (|\nu_m| \simeq k_0 a)$.

$$K\mathbf{\Psi}_m = \nu_m\Gamma\mathbf{\Psi}_m$$

$$\mathbf{\Psi}_m = \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} \qquad (4)$$

$$K = \begin{pmatrix} k_0 r\hat{\varepsilon} & -i\nabla_t \\ i\nabla_t & k_0 r I \end{pmatrix}$$

$$\nabla_t = \begin{pmatrix} 0 & -\dfrac{\partial}{\partial r}r & 0 \\ \dfrac{\partial}{\partial r}r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with the boundary conditions

$$[\mathbf{l}_r\mathbf{E}_m] = -\delta_e[\mathbf{l}_r[\mathbf{H}_m\mathbf{l}_r]]\Big|_{r=a}$$

$$\delta_e = \frac{1}{\sqrt{\varepsilon' - \sin^2\theta}} = \frac{1}{\sqrt{\varepsilon + i\sigma/\omega\varepsilon_0 - \sin^2\theta}}$$

θ is the incidence angle of the plain wave on the Earth surface; ε is the relative dielectric permittivity of the Earth surface, σ is its conductivity in S m^{-1}, and by the condition $\mathbf{\Psi}_m(r) \to 0$ at Im $k_0 > 0$ and $r \to \infty$, Γ have the form (2).

In real conditions (except the Antarctics, Greenland and the permafrost regions) $|\varepsilon'| \gg |\sin^2\theta|$, so δ_e with a high accuracy does not depend on the spectral parameter. Equation (4) is an equation for the eigenfunctions. In order to obtain the orthogonality relation for the eigenfunctions a scalar product is introduced and the conjugated operator K^+ is determined in the following way

$$(\mathbf{\Psi}_n^+, K\mathbf{\Psi}_m) = (K^+\mathbf{\Psi}_n^+, \mathbf{\Psi}_m) \qquad (5)$$

The vector standing at the first place is taken with the complex conjugation and the parenthesis mean a scalar product

$$(\mathbf{a}, \mathbf{b}) = \int_a^\infty \mathbf{a}^*\mathbf{b}rdr$$

Relation (5) takes place at

$$K^{+*} = \begin{pmatrix} k_0r\hat{\varepsilon}^T & i\nabla_t \\ -i\nabla_t & k_0rI \end{pmatrix} \qquad (5')$$

$\hat{\varepsilon}^T$ is the transposed tensor $\hat{\varepsilon}$ with the boundary conditions

$$[\mathbf{l}_r\mathbf{E}_m^+] = \delta_e^*[\mathbf{l}_r[\mathbf{H}_m^+\mathbf{l}_r]]\Big|_{r=a}$$

the sign $*$ designates a complex conjugation. The eigenfunctions of the adjoined operator satisfy the equation

$$K^+\mathbf{\Psi}_n^+ = \nu_n^*\Gamma\mathbf{\Psi}_n^+ \qquad (6)$$

the eigenvalues of equation (4) at the same indices coincide with the complex conjugated value of equation (6). We multiply the left-hand side of (4) to $r\mathbf{\Psi}_n^{+*}$, and the right-hand side of (6) to $r\mathbf{\Psi}_m$ and subtract the latter from the former. Then

$$r\mathbf{\Psi}_n^{+*}K\mathbf{\Psi}_m - K^{+*}\mathbf{\Psi}_n^{+*}r\mathbf{\Psi}_m = r(\nu_m - \nu_n)\mathbf{\Psi}_n^{+*}\Gamma\mathbf{\Psi}_m$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\{[r^2\mathbf{E}_n^{+*}\mathbf{H}_m]_r - [r^2\mathbf{H}_n^{+*}\mathbf{E}_m]_r\} =$$

$$i(\nu_m - \nu_n)\mathbf{\Psi}_n^{+*}\Gamma\mathbf{\Psi}_m$$

We integrate both sides of the latter equality forming a scalar product

$$\int_a^\infty \frac{\partial}{\partial r}\{[r^2\mathbf{E}_n^{+*}\mathbf{H}_m]_r - [r^2\mathbf{H}_n^{+*}\mathbf{E}_m]_r\}dr =$$

$$i(\nu_m - \nu_n)\int_a^\infty \mathbf{\Psi}_n^{+*}\Gamma\mathbf{\Psi}_m rdr = 0$$

$$\int_a^\infty \mathbf{\Psi}_n^{+*}\Gamma\mathbf{\Psi}_m rdr = (\mathbf{\Psi}_n^+, \Gamma\mathbf{\Psi}_m) = 2N_m\delta_{mn}$$

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

where

$$N_m = \frac{i}{2}\{[\mathbf{E}_m^{+*}\dot{\mathbf{H}}_m]_r - [\mathbf{H}_m^{+*}\dot{\mathbf{E}}_m]_r\}\cdot a^2\Big|_{r=a} \qquad (7)$$

$\dot{\mathbf{E}}_m = (\partial\mathbf{E}_m/\partial\nu_m)$ and $\dot{\mathbf{H}}_m = (\partial\mathbf{H}_m/\partial\nu_m)$ are derivatives with respect to the spectral parameter and N_m is the normalizing multiplier. Expression (3) for the solution we substitute into the initial equation (1) outside the sources region and multiply its left-hand side scalarly to $\mathbf{\Psi}_n^+$. Then we obtain

$$\frac{dA_n}{d\Theta} - i\nu_nA_n = -\frac{1}{2N_n}\sum A_m(\mathbf{\Psi}_n^+, \Gamma\frac{\partial\mathbf{\Psi}_m}{\partial\Theta}) \qquad (8)$$

The solution of the homogeneous equation (8) has an exponential dependence on the Θ coordinate. At the same time, it is known that in a regular waveguide, the angular dependence of normal waves is described by the Legendre function. Only while using asymptotic presentations of these functions in the wave zone relative to the source and its antipode, the Legendre function becomes approximately exponential. Therefore we obtain the limits of applicability of the solution (3) $|\cot\Theta| \ll |\nu_m|$ or $\Theta \gg (1/k_0a)$ and $\pi - \Theta \gg (1/k_0a)$, because $|\nu_m| \sim k_0a$.

[12] Assuming the eigenfunctions to be normalized, i.e., $N_n = 1$, rewrite equation (8) with respect to a new function $\Lambda_n(\Theta)$ outside the source area for a path segment, within which the waveguide characteristics are supposed constant. Let's denote Θ_0 the initial coordinate of such a segment. Assuming the eigenfunctions to be normalized, i.e., $N_n = 1$, we rewrite equation (8) with respect to a new function $\Lambda_n(\Theta)$ outside the sources for that part of the path within which the characteristics of the waveguide are considered to be constant. Let us designate the initial coordinate of such part as Θ_0.

$$A_n(\Theta) = \Lambda_n(\Theta)e^{i\int_{\Theta_0}^{\Theta}\nu_n(\Theta')d\Theta'} \qquad (9)$$

$$\frac{d\Lambda_n}{d\Theta} = \frac{1}{2}\sum\Lambda_m e^{i\int_{\Theta_0}^{\Theta}(\nu_m-\nu_n)d\Theta'}(\mathbf{\Psi}_n^+, \Gamma\frac{\partial\mathbf{\Psi}_m}{\partial\Theta}) \qquad (9')$$

The scalar product $(1/2)(\mathbf{\Psi}_n^+, \Gamma\partial\mathbf{\Psi}_m/\partial\Theta) = S_{mn}$ describes the differential matrix of transformation of normal waves [*Lutchenko et al.*, 1986]. In the real conditions, the electric properties of the ionosphere vary smoothly, so the S_{mn}

elements only due to the ionosphere would be continuous functions of the Θ argument. However variations in the conductivity of the lower boundary (in terms of the scales of field changes) in the accepted model occurs in a jump-like way.

[13] Let a jump-like change of the medium properties occurs in the cross section with a coordinate Θ_0. We designate $\tilde{\mathbf{\Phi}}^{(1)}$ field in the cross section $\Theta_0 - \varepsilon$ and $\tilde{\mathbf{\Phi}}^{(2)}$ in the cross section $\Theta_0 + \varepsilon$. The tangential components of the vectors $\Gamma\tilde{\mathbf{\Phi}}^{(1)} = \Gamma\tilde{\mathbf{\Phi}}^{(2)}$ should be continuous or

$$\sum A_m^{(1)}\Gamma\mathbf{\Psi}_m^{(1)} = \sum A_n^{(2)}\Gamma\mathbf{\Psi}_n^{(2)} \tag{10}$$

In the former sum, generally speaking, there should present positive and negative indices m ($m < 0$ corresponds to reflected waves). We multiply scalarly (10) to $\mathbf{\Psi}_n^{(2)+}$. Then

$$A_n^{(2)} = \frac{1}{2}\sum A_m^{(1)}(\mathbf{\Psi}_n^{(2)+}, \Gamma\mathbf{\Psi}_m^{(1)}) \tag{11}$$

In the obtained sum $A_m^{(1)}$, amplitudes of reflected waves $m < 0$ are unknown, however if we neglect them, the passed waves are determined in (11) and $A_n^{(2)}(\Theta_0) = \Lambda_n^{(2)}$.

[14] The scalar product

$$\frac{1}{2}(\mathbf{\Psi}_n^{(2)+}, \Gamma\mathbf{\Psi}_m^{(1)}) = \Pi_{nm} \tag{12}$$

may be interpreted as an element of the matrix Π of transformation of normal waves and the aggregate $\Lambda_n^{(2)}$ may be considered as the $\mathbf{\Lambda}^{(2)}$ vector which is determined by the product of the Π matrix to the $\mathbf{A}^{(1)}$ vector in the cross section Θ_0.

[15] The solution of system of differential equations ($9'$) presents a rather tiresome problem, so we will approximately take that the ionosphere is changing in a jump-like way. A sequence of cross sections is chosen along the path. In each cross section, characteristics of normal waves are determined and to the next cross section the waveguide is considered as homogeneous. At the joints of the cross sections, the matrix Π is calculated neglecting the reflected waves. Thus in the source vicinity, normal waves with the amplitudes $\Lambda_m^{(0)}$ are excited. At the next cross section located at the angular distance of $\Delta\Theta^{(0)}$ the waves will have amplitudes $A_m^{(0)} = \Lambda_m^{(0)} e^{i\nu_m^{(0)}\Delta\Theta^{(0)}}$. In the next cross section the amplitudes will be

$$\Lambda_n^{(1)} = \sum_m \Pi_{nm}^{(1)}\Lambda_m^{(0)} e^{i\nu_m^{(0)}\Delta\Theta^{(0)}}$$

and so on.

$$\sum_{k=0}^{N-1}\Delta\Theta^{(k)} = \Theta \qquad \Delta\Theta^{(k)} = \Theta^{(k+1)} - \Theta^{(k)}$$

$$\Lambda_j^{(N)} = \sum_i \Pi_{ji}^{(N)}\Lambda_i^{(N-1)} e^{i\nu_i^{(N-1)}\Delta\Theta^{(N-1)}} \tag{13}$$

We write the solution of system (1) in the form

$$\mathbf{\Phi}(r,\Theta) = \sum_m \mathbf{\Psi}_m^{(N)}(r)\Lambda_m^{(N)} e^{i\nu_m^{(N)}(\Theta-\Theta^{(N)})} \tag{14}$$

We determine the coefficients of normal wave excitation by a vertical dipole assuming that in the vicinity of the dipole $\Theta \leq (\lambda/a)$ the waveguide is homogeneous and so we use the regular waveguide model. The methods of calculation of the excitation coefficients are described below.

4. Method of Calculation of Normal Wave Characteristics for Waveguides of Comparison

[16] Over a homogeneous spherical surface $r > a$ there is a medium described by the dielectric permittivity tensor $\hat{\varepsilon}(r)$.

$$\hat{\varepsilon}(r) = I - \frac{X}{U(U^2 - Y^2)}\hat{M}$$

$$X = \frac{\omega_{pl}^2}{\omega^2} \qquad U = 1 + \frac{i\nu_e(r)}{\omega} \qquad \mathbf{Y} = \frac{|q_e|}{m_e\cdot\omega}\cdot\mathbf{B}_E$$

$$\omega_{pl}^2 = \frac{q_e^2 N_e}{m_e\varepsilon_0}$$

$$\widehat{M} =$$

$$\begin{pmatrix} U^2 - Y_\Theta^2 & -iY_rU - Y_\Theta Y_\varphi & iY_\varphi U - Y_\Theta Y_r \\ iY_rU - Y_\Theta Y_\varphi & U^2 - Y_\varphi^2 & -iY_\Theta U - Y_\varphi Y_r \\ -iY_\varphi U - Y_\Theta Y_r & iY_\Theta U - Y_\varphi Y_r & U^2 - Y_r^2 \end{pmatrix}$$

ν_e is the collision frequency of electrons with neutral particles, N_e is the electron concentration, $\mathbf{B}_E$ is the Earth magnetic field vector, $\mathbf{Y}$ is the relative gyrofrequency of electrons in the constant magnetic field of the Earth, ω is the angular frequency, and ω_{pl} is the plasma frequency. One has to solve the problem to the eigenvalues, i.e., equation (4).

[17] The initial system of equations consists of 6 scalar equations, however not all of them being differential ones. Using two equations one can express the vertical components of the vectors E_{mr} and H_{mr}

$$E_{mr} = -\frac{1}{\varepsilon_{rr}}\left(\frac{\nu_m}{k_0 r}H_{m\varphi} + \varepsilon_{r\Theta}E_{m\Theta} + \varepsilon_{r\varphi}E_{m\varphi}\right)$$

$$H_{mr} = \frac{\nu_m}{k_0 r}E_{m\varphi} \tag{15}$$

and obtain a new system of four differential equations

$$\begin{cases} \dfrac{d\mathbf{e}_1}{dk_0 r} = i(B\mathbf{e}_1 + C\mathbf{e}_2) \\ \dfrac{d\mathbf{e}_2}{dk_0 r} = i(D\mathbf{e}_1 + T\mathbf{e}_2) \end{cases} \qquad (16)$$

$$\mathbf{e}_1 = (H_{m\varphi}, -E_{m\varphi})^T$$

$$\mathbf{e}_2 = (E_{m\Theta}, H_{m\Theta})^T$$

$$B = \begin{pmatrix} -\dfrac{\varepsilon_{\Theta r}}{\varepsilon_{rr}}\dfrac{\nu_m}{k_0 r} + \dfrac{i}{k_0 r} & \dfrac{\varepsilon_{\Theta r}\varepsilon_{r\varphi}}{\varepsilon_{rr}} - \varepsilon_{\Theta\varphi} \\ 0 & \dfrac{i}{k_0 r} \end{pmatrix}$$

$$C = \begin{pmatrix} \varepsilon_{\Theta\Theta} - \dfrac{\varepsilon_{\Theta r}\varepsilon_{r\Theta}}{\varepsilon_{rr}} & 0 \\ 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 - \dfrac{\nu_m^2}{(k_0 r)^2}\dfrac{1}{\varepsilon_{rr}} & \dfrac{\nu_m}{k_0 r}\dfrac{\varepsilon_{r\varphi}}{\varepsilon_{rr}} \\ \dfrac{\nu_m}{k_0 r}\dfrac{\varepsilon_{\varphi r}}{\varepsilon_{rr}} & \varepsilon_{\varphi\varphi} - \dfrac{\varepsilon_{r\varphi}\varepsilon_{\varphi r}}{\varepsilon_{rr}} - \dfrac{\nu_m^2}{(k_0 r)^2} \end{pmatrix}$$

$$T = \begin{pmatrix} \dfrac{i}{k_0 r} - \dfrac{\nu_m}{k_0 r}\dfrac{\varepsilon_{r\Theta}}{\varepsilon_{rr}} & 0 \\ \dfrac{\varepsilon_{r\Theta}\varepsilon_{\varphi r}}{\varepsilon_{rr}} - \varepsilon_{\varphi\Theta} & \dfrac{i}{k_0 r} \end{pmatrix}$$

[18] It is known that a numerical integration of system (16) requires special approaches, because one has to separate two solutions satisfying the condition of a decrease at the infinity out of four independent solutions available. At the proper specification of the initial conditions at a height of r_u, (the choice of which was described in section 2) and integration from the top downward, we get rid of two undesirable solutions, however out of two solutions left, one will very quickly increase at the integration. To obtain a solution stable to the integration errors, we use a bivector formed from two solutions

$$W_{ik} = b_i^{(1)} b_k^{(2)} - b_i^{(2)} b_k^{(1)} \qquad i, k = 1, 2, 3, 4 \qquad (17)$$

where

$$\mathbf{b} = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \qquad \mathbf{b} = (H_{m\varphi}, -E_{m\varphi}, E_{m\Theta}, H_{m\Theta})^T$$

$$\frac{d\mathbf{b}}{dk_0 r} = iF\mathbf{b} \qquad \text{where } F = \begin{pmatrix} B & C \\ D & T \end{pmatrix}$$

j is the number of the solution of system (16) $j = 1, 2$.

[19] The bivector is a skew-symmetric tensor 4×4. At its diagonal there are zeros and $W_{ik} = -W_{ki}$, so it has 6 independent elements. The bivector satisfies the following system of equations:

$$\frac{dW}{dk_0 r} = i(FW + WF^T) \qquad (18)$$

[20] We determine the initial values W_{ik} at a height r_u from the solution of the problem on the field of a plain wave in the longitudinal approximation

$$\mathbf{e}_1^{(1)} = \begin{pmatrix} \dfrac{h^{(1)}}{k_0} \\ i \end{pmatrix} \qquad \mathbf{e}_2^{(1)} = \begin{pmatrix} 1 \\ i\dfrac{h^{(1)}}{k_0} \end{pmatrix}$$

$$\mathbf{e}_1^{(2)} = \begin{pmatrix} \dfrac{h^{(2)}}{k_0} \\ -i \end{pmatrix} \qquad \mathbf{e}_2^{(2)} = \begin{pmatrix} 1 \\ -i\dfrac{h^{(2)}}{k_0} \end{pmatrix}$$

$$\frac{h^{(1)}}{k_0} = \sqrt{\frac{X}{Y_r - U}} \qquad \frac{h^{(2)}}{k_0} = i\sqrt{\frac{X}{Y_r + U}}$$

Using the bivector elements, one can obtain the matrix of the reflective properties of the ionosphere $\hat{\alpha}$ which is determined by the relation

$$\mathbf{e}_1 = \hat{\alpha}\mathbf{e}_2 \qquad (19)$$

$$\alpha_{11} = \frac{W_{14}}{W_{34}} \qquad \alpha_{12} = -\frac{W_{13}}{W_{34}}$$

$$\alpha_{21} = \frac{W_{24}}{W_{34}} \qquad \alpha_{22} = -\frac{W_{23}}{W_{34}} \qquad (20)$$

The system of equations for the $\hat{\alpha}$ matrix is nonlinear, so at its integration, singularities may arise (W_{34} may reduce to zero). Such phenomena arise sometimes for the equatorial ionosphere. Two points are merits of this new method of bivector: the linearity of system (18) and the fact that at no conditions the W_{ik} elements transform to the infinity.

[21] At the r_u level, elements of the $\hat{\alpha}$ matrix do not depend on the spectral parameter, so

$$\int_{r_u}^{\infty} \mathbf{\Psi}_m^{+*} \cdot \Gamma \mathbf{\Psi}_m r dr = 0$$

Therefore it follows that the scalar products $(\mathbf{\Psi}_m^+, \Gamma\mathbf{\Psi}_m)$, determined at the intervals $[a, \infty)$ and $[a, r_u]$, coincide.

[22] The characteristic equation for the eigenvalues of the waveguide problem is obtained from the boundary conditions at the Earth surface

$$f(\nu_m) = (1 + \alpha_{11}\delta_e)(\delta_e + \alpha_{22}) + \alpha_{12}\alpha_{21}\delta_e = 0 \qquad (21)$$

The eigenvalues ν_m are included into the coefficients of equations (18); therefore the values of the matrix $\hat{\alpha}$ elements at the Earth surface depend on these values. To determine ν_m, the Newton method

$$f(\nu_m) = f(\nu_m^0) + \frac{\partial f}{\partial \nu_m}(\nu_m - \nu_m^0) \qquad (22)$$

is used, where ν_m^0 is the initial approximation for the eigenvalue. Then

$$\nu_m - \nu_m^0 = -\frac{f(\nu_m^0)}{\partial f/\partial \nu}$$

[23] Numerical realization of this algorithm and the procedure of the selection of initial approximations for eigenvalues are discussed in the next paragraph.

[24] At the Earth surface, the derivatives of the $\hat{\alpha}$ matrix elements with respect to the spectral parameter $\partial\hat{\alpha}/\partial\nu$ are usually large, so presentation (22) is correct at small $\nu_m - \nu_m^0$. If the initial approximation is given with insufficient accuracy, then to obtain the eigenvalue one would need many iterations, and at each iteration system (18) should be integrated together with the system for the derivatives with respect to the spectral parameter $\dot{W} = \partial W/\partial \nu$

$$\frac{d\dot{W}}{dk_0 r} = i(\dot{F}W + F\dot{W} + \dot{W}F^T + W\dot{F}^T) \qquad (23)$$

To rush the iteration process, the following approach is used: functions satisfying the boundary conditions at the Earth surface are constructed in the vacuum spherical cavity, and the characteristic equation is obtained from equation system (19) which is valid at any height r_{ef} under the condition that at $r < r_{\mathrm{ef}}$ vacuum, i.e. $\mathbf{e}_1$ and $\mathbf{e}_2$ are obtained for the vacuum. It is convenient to chose such height r_{ef}, where $\hat{\alpha}$ is least dependent on the spectral parameter. We will call it an effective height. We chose r_{ef} according to the criterion $\min \sum \mid \dot{\alpha}_{ik} \mid$. Thus, calculating the eigenvalues, systems (18) and (23) are integrated from r_u to r_l (see section 2). Then the same systems of equations, but with the coefficients corresponding to the vacuum $\varepsilon_{ik} = 0$ and $\varepsilon_{ii} = 1$ are integrated upward till the condition $\min \sum \mid \dot{\alpha}_{ik} \mid$ is fulfilled.

[25] For realization of the above described scheme of recalculation of the $\hat{\alpha}$ matrix to the effective height r_{ef}, a solution should be formed in the spherical cavity in the vacuum at the given boundary conditions of an impedance type at the upper and lower walls.

[26] Following *Makarov et al.* [1991] and *Kirillov* [1979], we consider the electromagnetic field independent of the azimuthal coordinate φ. We describe the field using the pair of the Hertz potentials U_1 and U_2, satisfying in the cavity $a \le r \le r_{\mathrm{ef}}$ to the following system of differential equations:

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2 \sin\Theta}\frac{\partial}{\partial\Theta}\sin\Theta\frac{\partial}{\partial\Theta} + k_0^2\Big)\mathbf{U} =$$

$$\begin{pmatrix} -j_r^e\sqrt{\dfrac{\mu_o}{\varepsilon_o}} \\ \\ 0 \end{pmatrix} \qquad (24)$$

[27] The electromagnetic field is related to the potential by the formulae

$$\tilde{\mathcal{E}}_r = -\frac{1}{ik_0 r^2 \sin\Theta}\frac{\partial}{\partial\Theta}\sin\Theta\frac{\partial}{\partial\Theta}U_1 - \frac{j_r^e}{i\omega\varepsilon_0}$$

$$\tilde{\mathcal{E}}_\Theta = \frac{1}{ik_0 r}\frac{\partial^2}{\partial r\partial\Theta}U_1$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\tilde{\mathcal{H}}_\varphi = \frac{1}{r}\frac{\partial}{\partial\Theta}U_1$$

$$-\tilde{\mathcal{E}}_\varphi = \frac{1}{r}\frac{\partial}{\partial\Theta}U_2$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\tilde{\mathcal{H}}_\Theta = \frac{1}{ik_0 r}\frac{\partial^2}{\partial r\partial\Theta}U_2$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\tilde{\mathcal{H}}_r = -\frac{1}{ik_0 r^2 \sin\Theta}\frac{\partial}{\partial\Theta}\sin\Theta\frac{\partial}{\partial\Theta}U_2$$

[28] The conditions of the impedance type for the potentials have the following form: at $r = a$

$$\Big(ik_0\delta_e + \frac{\partial}{\partial r}\Big)U_1 = 0 \qquad \Big(1 + \frac{\delta_e}{ik_0}\frac{\partial}{\partial r}\Big)U_2 = 0 \qquad (25)$$

and at $r = r_{\mathrm{ef}}$

$$\Big(1 - \frac{\hat{\alpha}}{ik_0}\frac{\partial}{\partial r}\Big)\mathbf{U} = 0 \qquad \mathbf{U} = (U_1, U_2)^T \qquad (26)$$

We solve the system of differential equations (24) by the method of separation of variables with expansion in terms of normal waves

$$\mathbf{U}(r,\Theta) = \sum_m \tilde{\Lambda}_m \mathbf{R}_m(r)\Theta_m(\Theta) \qquad (27)$$

$$\frac{1}{\sin\Theta}\frac{d}{d\Theta}\sin\Theta\frac{d}{d\Theta}\Theta_m + (\nu_m^2 - 1/4)\Theta_m =$$

$$-\frac{\delta(\Theta)}{2\pi\sin\Theta} \qquad (28)$$

$$\Big(\frac{d^2}{dr^2} + k_0^2 - \frac{\nu_m^2 - 1/4}{r^2}\Big)\mathbf{R}_m = 0$$

Substituting (27) into equation (24) and taking into account (28), we obtain

$$\sum_m \tilde{\Lambda}_m \mathbf{R_m} = -\mathbf{B}_0 \delta(r-b) \qquad (28')$$

where

$$\mathbf{B}_0 = \begin{pmatrix} \sqrt{\dfrac{\mu_0}{\varepsilon_0}} J l_p \\ 0 \end{pmatrix}$$

at the excitation of the field by a short antenna. The δ function in the right-hand sides of equations (28), (28′) appears because the source of the field is a point dipole situated at the height $b-a$ above the Earth's surface.

[29] The function

$$\Theta_m(\Theta) = \frac{1}{4\cos(\nu_m \pi)} P_{\nu_m - 1/2}(-\cos\Theta)$$

in the interval

$$\frac{1}{|\nu_m|} < \Theta < \pi - \frac{1}{|\nu_m|}$$

is a limited solution of equation (28′) at $\Theta = \pi$. The function $\Theta_m(\Theta)$ may be presented in the form

$$\Theta_m(\Theta) \cong \frac{1}{2}\sqrt{\frac{1}{2\pi\nu_m \sin\Theta}} e^{i\nu_m\Theta + i\dfrac{\pi}{4}} \qquad (29)$$

where we neglected the antipode wave and around-the-globe waves. To obtain a solution bounded at $\Theta = 0$ the source, one should take into account the finite thickness of the antenna. We present it as a hollow short truncated cone the outer surface of which is described by $\Theta = \varepsilon$. Further on we should construct a solution bounded in the region $\Theta \le \varepsilon$, to sew solutions in the regions $\Theta > \varepsilon$ and $\Theta \le \varepsilon$ at the boundary $\Theta = \varepsilon$ in the way it is described in detail by *Makarov et al.* [1991]. Since we are interested in the distance much longer than the antenna thickness, ε may be tended to zero, obtaining formula (29). We introduce new functions $\mathbf{V}_m$, and $\mathbf{R}_m = \sqrt{(r/b)}\mathbf{V}_m(r)$ and new variable

$$\xi = \frac{k_0 r_{\rm ef}}{M} \ln(\frac{r_{\rm ef}}{r})$$

such that

$$r = r_{\rm ef} \exp(-\frac{\xi M}{k_0 r_{\rm ef}}) \qquad M^3 = \frac{k_0 r_{\rm ef}}{2}$$

Neglecting by the terms of the order of $((r_{\rm ef} - a)/a)^2$ and $M/k_0 r_{\rm ef}$ we obtain the equation with the boundary conditions

$$\big(\lambda - \xi + \frac{d^2}{d\xi^2}\big)\mathbf{V} = 0 \qquad (30)$$

$$\big(1 - \tau\frac{d}{d\xi}\big)\mathbf{V} = 0 \qquad \xi = 0 \text{ at } r = r_{\rm ef} \qquad (31)$$

$$\big(\hat{t_1} + \hat{t_2}\frac{d}{d\xi}\big)\mathbf{V} = 0 \qquad \xi = l \text{ at } r = a \qquad (32)$$

where

$$\tau = \frac{i\hat{\alpha}}{M}$$

$$\hat{t_1} = {\rm diag}\{t_1, 1\} \qquad \hat{t_2} = {\rm diag}\{1, t_2\}$$

$$l = \frac{k_0 r_{\rm ef}}{M}\ln(\frac{r_{\rm ef}}{a})$$

$$t_1 = -iM\delta_1 \qquad t_2 = \frac{i\delta_2}{M}$$

$$\delta_1 = \frac{a}{r_{\rm ef}}\delta_e - \frac{i}{2k_0 r_{\rm ef}} \qquad \delta_2 \simeq \frac{r_{\rm ef}}{a}\delta_e$$

The vector function $\mathbf{V}$ and spectral parameter λ are used at the formulation of the problem without indices, $\mathbf{V}_m$ is the eigenvector function, and λ_m is the eigenvalue. The eigennumber ν_m is related to the new spectral parameter λ_m by the relation $\nu_m = k_0 r_{\rm ef}\sqrt{1 - (\lambda_m/M^2)}$.

[30] Complex conjugate values of the eigenfunctions of the adjoint operator $\mathbf{V}_m^{+*}$ satisfy equation (30) with the same boundary conditions but only with the transposed matrix τ. We multiply equation (30) with the spectral parameter λ_m and function $\mathbf{V}_m$ to the function $\mathbf{V}_n^{+*}$, satisfying equation (30) with the spectral parameter λ_n. We multiply equation (30) to $\mathbf{V}_m$ and subtract the latter from the former. Then

$$(\lambda_m - \lambda_n)\mathbf{V}_m\mathbf{V}_n^{+*} =$$

$$\mathbf{V}_m \frac{d^2}{d\xi^2}\mathbf{V}_n^{+*} - \mathbf{V}_n^{+*}\frac{d^2}{d\xi^2}\mathbf{V}_m \qquad (33)$$

We integrate (33) with respect to ξ within the spherical cavity $(0, l)$ and obtain

$$(\lambda_m - \lambda_n)\int_0^l \mathbf{V}_n^{+*}\mathbf{V}_m d\xi =$$

$$\mathbf{V}_m \frac{d}{d\xi}\mathbf{V}_n^{+*}\Big|_{\xi=l} - \mathbf{V}_n^{+*}\frac{d}{d\xi}\mathbf{V}_m\Big|_{\xi=l} -$$

$$\mathbf{V}_m \frac{d}{d\xi}\mathbf{V}_n^{+*}\Big|_{\xi=0} + \mathbf{V}_n^{+*}\frac{d}{d\xi}\mathbf{V}_m\Big|_{\xi=0} \qquad (34)$$

If the boundary condition for $\mathbf{V}_m$ and $\mathbf{V}_n^{+*}$ at $\xi = l$ and $\xi = 0$ does not depend on the spectral parameter, then the right-hand side of (34) tends to 0 and the functions $\mathbf{V}_m$ and $\mathbf{V}_n^{+*}$ form an orthogonal system.

[31] To construct the eigenfunctions $\mathbf{V}_m(\xi)$ we consider two independent solutions of equation (30) $F_{i0}(\lambda, \xi)$ $(i =$

0, 1) and their derivatives $F_{i1}(\lambda,\xi)$. They in pairs satisfy the following system

$$\frac{\partial F_{i0}}{\partial \xi} = F_{i1} \qquad \frac{\partial F_{i1}}{\partial \xi} = (\xi-\lambda)F_{i0} \tag{35}$$

with the initial conditions

$$F_{00}(\lambda,0)=0 \qquad F_{10}(\lambda,0)=-1$$

$$F_{01}(\lambda,0)=1 \qquad F_{11}(\lambda,0)=0 \tag{36}$$

Functions $F_{i0}(\lambda,\xi)$ are the Airy's functions and the Jacobian composed of two solutions $F_{i0}(\lambda,\xi)$, is independent of ξ

$$F_{00}F_{11}-F_{01}F_{10}=1 \tag{37}$$

As functions of the λ parameter, $F_{ik}(\lambda,\xi)$ satisfy the following system of differential equations

$$\frac{\partial F_{00}}{\partial \lambda} = -F_{01}-F_{10}$$

$$\frac{\partial F_{10}}{\partial \lambda} = \lambda F_{00}-F_{11}$$

$$\frac{\partial F_{01}}{\partial \lambda} = -(\xi-\lambda)F_{00}-F_{11}$$

$$\frac{\partial F_{11}}{\partial \lambda} = \lambda F_{01}-(\xi-\lambda)F_{10} \tag{38}$$

System (38) makes it possible to continue analytically in terms of λ the aggregate of the functions $F_{ik}(\lambda,\xi)$, if they are known at some λ. System (38) is obtained in the following way. Let $\tilde{U}_1(\xi)$ and $\tilde{U}_2(\xi)$ be 2 independent solutions of the Airy's equation

$$\frac{d^2\tilde{U}}{d\xi^2}-\xi\tilde{U}=0$$

Then the system of equations (35) is satisfied if one takes

$$F_{00}(\lambda,\xi)=\tilde{U}_1(-\lambda)\tilde{U}_2(\xi-\lambda)-\tilde{U}_2(-\lambda)\tilde{U}_1(\xi-\lambda)$$

$$F_{00}(\lambda,0)=0$$

$$F_{01}(\lambda,\xi)=\tilde{U}_1(-\lambda)\tilde{U}_2'(\xi-\lambda)-\tilde{U}_2(-\lambda)\tilde{U}_1'(\xi-\lambda)$$

$$F_{01}(\lambda,0)=1$$

$$F_{10}(\lambda,\xi)=\tilde{U}_1'(-\lambda)\tilde{U}_2(\xi-\lambda)-\tilde{U}_2'(-\lambda)\tilde{U}_1(\xi-\lambda)$$

$$F_{10}(\lambda,0)=-1$$

$$F_{11}(\lambda,\xi)=\tilde{U}_1'(-\lambda)\tilde{U}_2'(\xi-\lambda)-\tilde{U}_2'(-\lambda)\tilde{U}_1'(\xi-\lambda)$$

$$F_{11}(\lambda,0)=0 \tag{38$'$}$$

where

$$\tilde{U}'(\xi)=\frac{d\tilde{U}(\xi)}{d\xi}$$

Thus, relations (35), (36), and (37) are satisfied. We differentiate (38) with respect to λ and obtain system (38′).

The vector-function $\mathbf{V}(\lambda,\xi)$ and its derivative $[d\mathbf{V}/d\xi](\lambda,\xi)$ are expressed via $F_{ik}(\lambda,\xi)$ in the following way

$$\mathbf{V}(\lambda,\xi)=F_{00}(\lambda,\xi)\mathbf{x}+F_{10}(\lambda,\xi)\mathbf{y} \tag{39}$$

$$\frac{d\mathbf{V}}{d\xi}(\lambda,\xi)=F_{01}(\lambda,\xi)\mathbf{x}+F_{11}(\lambda,\xi)\mathbf{y} \tag{40}$$

where vectors $\mathbf{x}$ and $\mathbf{y}$ are independent of ξ. Using the initial values (36) at the upper wall of the waveguide ($\xi=0$), we obtain

$$\mathbf{V}(\lambda,0)=-\mathbf{y} \qquad \frac{d\mathbf{V}}{d\xi}(\lambda,0)=\mathbf{x}$$

The boundary conditions are satisfied if

$$\mathbf{y}=-\tau\mathbf{x}$$

At the lower wall of the waveguide under $\xi=l$, the boundary conditions are satisfied if

$$\mathbf{x}=-T\mathbf{y}$$

where

$$T=\mathrm{diag}[T_1,T_2]$$

$$T_1=D_{11}/D_{01}$$

$$T_2=D_{10}/D_{00}$$

$$D_{00}=F_{00}(\lambda,l)+t_2F_{01}(\lambda,l)$$

$$D_{01}=F_{01}(\lambda,l)+t_1F_{00}(\lambda,l)$$

$$D_{10}=F_{10}(\lambda,l)+t_2F_{11}(\lambda,l)$$

$$D_{11}=F_{11}(\lambda,l)+t_1F_{10}(\lambda,l) \tag{40$'$}$$

Here $t_1=-iM\delta_1$ and $t_2=i\delta_2/M$. To fulfill the conditions at the lower and upper boundaries, the equality should be valid

$$\mathbf{x}=T\tau\mathbf{x} \qquad \mathbf{x}=(x_1,x_2)^T \tag{41}$$

Equation system (41) relative x_1 and x_2 is linear and homogeneous. For the existence of a nonzero solution, the determinant should be made to vanish.

$$f(\lambda_m) = [1 - T_1\tau_{11}][1 - T_2\tau_{22}] - T_1\tau_{12}T_2\tau_{21} = 0 \qquad (42)$$

Equation (42) may be considered as characteristic equation recalculated from the Earth surface to the effective height, and its roots are the eigenvalues λ_m. The T and τ matrices are functions of λ_m.

[32] The roots of equation (42) are looked for by the iteration method, the convergence of the process depends on the accuracy of taking the initial approximations λ^0. The method of calculation of the initial approximations was developed by *Kirillov* [1979, 1981, 1983]. This method is based on the utilization of eigenvalues corresponding to the waveguide with ideally conducting lower boundary and magnetic upper boundary by insertion of corrections that take into account finite values of electric and magnetic parameters of the boundaries. The correction to the spectral parameter λ is found in the following way

$$\lambda - \lambda^0 = -\frac{f(\lambda^0)}{(\partial f/\partial\lambda)} \qquad (43)$$

where

$$\frac{\partial f}{\partial\lambda} = -(\dot{T}_1\tau_{11} + T_1\dot{\tau}_{11})(1 - T_2\tau_{22}) -$$

$$(\dot{T}_2\tau_{22} + T_2\dot{\tau}_{22})(1 - T_1\tau_{11}) -$$

$$\dot{T}_1\tau_{12}T_2\tau_{21} - T_1\dot{\tau}_{12}T_2\tau_{21} -$$

$$T_1\tau_{12}\dot{T}_2\tau_{21} - T_1\tau_{12}T_2\dot{\tau}_{21}$$

$$\dot{T}_1 = \frac{\dot{D}_{11}}{D_{01}} - \frac{D_{11}\dot{D}_{01}}{(D_{01})^2}$$

$$\dot{T}_2 = \frac{\dot{D}_{10}}{D_{00}} - \frac{D_{10}\dot{D}_{00}}{(D_{00})^2}$$

$$\dot{D}_{00} = -F_{01} - F_{10} + t_2[(\lambda - l)F_{00} - F_{11}]$$

$$\dot{D}_{01} = (\lambda - l)F_{00} - F_{11} + t_1[-F_{01} - F_{10}]$$

$$\dot{D}_{10} = \lambda F_{00} - F_{11} + t_2[\lambda F_{01} - (l - \lambda)F_{10}]$$

$$\dot{D}_{11} = \lambda F_{01} - (l - \lambda)F_{10} + t_2[\lambda F_{00} - F_{11}]$$

$$\dot{\tau} = i\hat{\alpha}/M$$

Here dot denotes a differentiation with respect to the spectral parameter λ. The found eigenvalues are excluded out of characteristic equation (42) while looking for the next values. Let λ_1 is found. Then the characteristic equation may be presented in the form

$$f(\lambda) = f_1(\lambda)\,(\lambda - \lambda_1)$$

and we obtain the corrections to the eigenvalue by the formula

$$(\lambda_2 - \lambda_2^0) = -\frac{f_1(\lambda_2^0)}{\dfrac{\partial f_1}{\partial\lambda}} = -\frac{f(\lambda_2^0)}{\dfrac{\partial f}{\partial\lambda} - \dfrac{f(\lambda_2^0)}{\lambda_2^0 - \lambda_1}}$$

Let n eigenvalues are found. Then looking for the n+1 value, we use the formula

$$\lambda_{n+1} - \lambda_{n+1}^0 =$$

$$-\frac{f(\lambda_{n+1}^0)}{\dfrac{\partial f(\lambda_{n+1}^0)}{\partial\lambda} - f(\lambda_{n+1}^0)\displaystyle\sum_{k=1}^{n}\frac{1}{(\lambda_{n+1}^0 - \lambda_k)}}$$

For determination of the excitation coefficients or amplitudes of the normal waves, we rewrite equation (28′) in terms of the introduced functions $\mathbf{V}_m$ of the variable ξ

$$\sum_m \tilde{\Lambda}_m\mathbf{V}_m = -\mathbf{P}_0\frac{k_0 r_{\text{ef}}}{M\cdot b}\delta(\xi - \xi_b) \qquad (44)$$

where

$$M = (\frac{k_0 r_{\text{ef}}}{2})^{1/3}$$

$$\xi_b = \frac{k_0 r_{\text{ef}}}{M}\ln(\frac{r_{\text{ef}}}{b})$$

$$\mathbf{V}_m = (V_{1m}, V_{2m})^T$$

We multiply scalarly the left-hand side of equation (44) to the eigenfunction of the adjoint operator $\mathbf{V}_m^+$ and, taking them orthogonal at the interval $[0, l]$, we obtain the excitation coefficient of the field by a short vertical antenna

$$\tilde{\Lambda}_m = -Jl_p\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{k_0}{M}\,\frac{r_{\text{ef}}}{b}\mathbf{V}_{1m}^{+*}(\xi_b) \qquad (45)$$

We write out the components of the electromagnetic field in the vacuum cavity of a regular waveguide

$$\tilde{\mathcal{E}}_\Theta(r,\Theta) = \sum_m \tilde{\Lambda}_m\frac{1}{2}\frac{e^{i\pi/4 + i\nu_m\Theta}}{\sqrt{2\pi\nu_m}\sin\Theta}\,\frac{\nu_m}{k_0 r\sqrt{rb}}\times$$

$$\left[\frac{1}{2}V_{1m}(\xi) - \frac{k_0 r_{\text{ef}}}{M}\frac{dV_{1m}(\xi)}{d\xi}\right] \qquad (46)$$

$$\tilde{\mathcal{E}}_\varphi(r,\Theta) =$$

$$-i\sum_m \tilde{\Lambda}_m \frac{1}{2}\frac{e^{i\pi/4+i\nu_m\Theta}}{\sqrt{2\pi\nu_m\sin\Theta}}\frac{\nu_m}{\sqrt{rb}}V_{2m}(\xi) \tag{47}$$

$$\tilde{\mathcal{E}}_r(r,\Theta) =$$

$$-i\sum_m \tilde{\Lambda}_m \frac{1}{2}\frac{e^{i\pi/4+i\nu_m\Theta}}{\sqrt{2\pi\nu_m\sin\Theta}}\frac{\nu_m^2}{k_0r\sqrt{rb}}V_{1m}(\xi) \tag{48}$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\tilde{\mathcal{H}}_\Theta(r,\Theta) = \sum_m \tilde{\Lambda}_m \frac{1}{2}\frac{e^{i\pi/4+i\nu_m\Theta}}{\sqrt{2\pi\nu_m\sin\Theta}}\frac{\nu_m}{k_0r\sqrt{rb}}\times$$

$$\Big[\frac{1}{2}V_{2m}(\xi) - \frac{k_0r_{\rm ef}}{M}\frac{dV_{2m}(\xi)}{d\xi}\Big] \tag{49}$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\tilde{\mathcal{H}}_\varphi(r,\Theta) =$$

$$i\sum_m \tilde{\Lambda}_m \frac{1}{2}\frac{e^{i\pi/4+i\nu_m\Theta}}{\sqrt{2\pi\nu_m\sin\Theta}}\frac{\nu_m}{\sqrt{rb}}V_{1m}(\xi) \tag{50}$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\tilde{\mathcal{H}}_r(r,\Theta) =$$

$$-i\sum_m \tilde{\Lambda}_m \frac{1}{2}\frac{e^{i\pi/4+i\nu_m\Theta}}{\sqrt{2\pi\nu_m\sin\Theta}}\frac{\nu_m^2}{k_0r\sqrt{rb}}V_{2m}(\xi) \tag{51}$$

5. Main Formulae for the Eigenfunctions, Excitation Coefficients, and the Field at the Observational Point

[33] We write down the normalized elements of six vectors $\mathbf{\Psi}_m$ (eigenfunctions of the lateral operator K) in the vacuum cavity, neglecting the terms of the order of $1/|\nu_m|$.

$$E_{m\Theta}(r) = \frac{\nu_m}{k_0r\sqrt{rb}}\big[\frac{1}{2}V_{1m}(\xi) - \frac{k_0r_{\rm ef}}{M}\frac{dV_{1m}(\xi)}{d\xi}\big]\sqrt{\frac{1}{a_m}} \tag{52}$$

$$E_{m\varphi}(r) = -\frac{i\nu_m}{\sqrt{rb}}V_{2m}(\xi)\frac{1}{\sqrt{a_m}} \tag{53}$$

$$E_{mr}(r) = \frac{\nu_m^2}{ik_0r\sqrt{rb}}V_{1m}(\xi)\frac{1}{\sqrt{a_m}} \tag{54}$$

$$H_{m\Theta}(r) = \frac{\nu_m}{k_0r}\frac{1}{\sqrt{rb}}\big[\frac{1}{2}V_{2m}(\xi) -$$

$$\frac{k_0r_{\rm ef}}{M}\frac{dV_{2m}(\xi)}{d\xi}\big]\frac{1}{\sqrt{a_m}} \tag{55}$$

$$H_{m\varphi}(r) = \frac{i\nu_m}{\sqrt{rb}}V_{1m}(\xi)\frac{1}{\sqrt{a_m}} \tag{56}$$

$$H_{mr}(r) = \frac{\nu_m^2}{ik_0r\sqrt{rb}}V_{2m}(\xi)\frac{1}{\sqrt{a_m}} \tag{57}$$

and elements Ψ_m^{+*} of the eigenfunctions of the adjoint operator K^{+*}

$$E_{m\Theta}^{+*}(r) = \frac{\nu_m}{k_0r\sqrt{rb}}\big[\frac{1}{2}V_{1m}^{+*}(\xi) -$$

$$\frac{k_0r_{\rm ef}}{M}\frac{dV_{1m}^{+*}(\xi)}{d\xi}\big]\sqrt{\frac{1}{a_m}}$$

$$E_{m\varphi}^{+*}(r) = -\frac{i\nu_m}{\sqrt{rb}}V_{2m}^{+*}(\xi)\frac{1}{\sqrt{a_m}}$$

$$E_{mr}^{+*}(r) = \frac{\nu_m^2}{ik_0r\sqrt{rb}}V_{1m}^{+*}(\xi)\frac{1}{\sqrt{a_m}}$$

$$H_{m\Theta}^{+*}(r) = \frac{\nu_m}{k_0r}\frac{1}{\sqrt{rb}}\big[\frac{1}{2}V_{2m}^{+*}(\xi) -$$

$$\frac{k_0r_{\rm ef}}{M}\frac{dV_{2m}^{+*}(\xi)}{d\xi}\big]\frac{1}{\sqrt{a_m}}$$

$$H_{m\varphi}^{+*}(r) = \frac{i\nu_m}{\sqrt{rb}}V_{1m}^{+*}(\xi)\frac{1}{\sqrt{a_m}}$$

$$H_{mr}^{+*}(r) = \frac{\nu_m^2}{ik_0r\sqrt{rb}}V_{2m}^{+*}(\xi)\frac{1}{\sqrt{a_m}}$$

$$a_m = -\nu_m^3/(2M^2k_0b) \qquad \mathbf{V}_m = (V_{1m}, V_{2m})^T$$

$$(\mathbf{V_m^+}, \mathbf{V_m}) = 1 \qquad (\mathbf{\Psi}_m^+, \Gamma\mathbf{\Psi}_m) = 2$$

$$M = \big(\frac{k_0r_{\rm ef}}{2}\big)^{1/3} \qquad \xi = \frac{k_0r_{\rm ef}}{M}ln\frac{r_{\rm ef}}{r}$$

We find the normalizing factor n_m with the help of formula (34), first differentiating it with respect to λ_1, and then equalizing $\lambda_1 = \lambda_2$.

$$n_m = \big(\frac{\partial F_{00}}{\partial \lambda} x_1 + \frac{\partial F_{10}}{\partial \lambda} y_1\big) \frac{dV_{1m}^{+*}}{d\xi}\big|_{\xi=l} +$$

$$\big(\frac{\partial F_{00}}{\partial \lambda} x_2 + \frac{\partial F_{10}}{\partial \lambda} y_2\big) \frac{dV_{2m}^{+*}}{d\xi}\big|_{\xi=l} -$$

$$\Big(\frac{\partial F_{10}}{\partial \lambda} x_1 + \frac{\partial F_{11}}{\partial \lambda} y_1\Big) V_{1m}^{+*}\big|_{\xi=l} -$$

$$\Big(\frac{\partial F_{10}}{\partial \lambda} x_2 + \frac{\partial F_{11}}{\partial \lambda} y_2\Big) V_{2m}^{+*}\big|_{\xi=l} \quad (58)$$

The derivatives of the function F_{ik} with respect to the spectral parameter λ are expressed via functions F_{ik} by formulae (38), the derivatives vanishing at $\xi = 0$. The eigenfunctions V satisfy the boundary conditions, therefore $y_1 = -(x_1/T_1)$, $y_2 = -(x_2/T_2)$, and $T_1 = (D_{11}/D_{01})$, $T_2 = D_{10}/D_{00})$. The eigenfunctions at any ξ are expressed by the following formulae

$$V_{1m}(\xi) = \big[F_{00}(\lambda_m \xi) \cdot D_{11} -$$

$$F_{10}(\lambda_m \xi) \cdot D_{01}\big] \frac{1}{\sqrt{n_m}} \frac{x_1}{D_{11}} \quad (59)$$

$$V_{2m}(\xi) = \big[F_{00}(\lambda_m \xi) \cdot D_{10} -$$

$$F_{10}(\lambda_m \xi) \cdot D_{00}\big] \frac{1}{\sqrt{n_m}} \frac{x_2}{D_{10}} \quad (60)$$

$$\frac{dV_{1m}(\xi)}{d\xi} = \big[F_{01}(\lambda_m \xi) \cdot D_{11} -$$

$$F_{11}(\lambda_m \xi) \cdot D_{01}\big] \frac{1}{\sqrt{n_m}} \frac{x_1}{D_{11}} \quad (61)$$

$$\frac{dV_{2m}(\xi)}{d\xi} = \big[F_{01}(\lambda_m \xi) \cdot D_{10} -$$

$$F_{10}(\lambda_m \xi) \cdot D_{00}\big] \frac{1}{\sqrt{n_m}} \frac{x_2}{D_{10}} \quad (62)$$

The functions $F_{ik}(\lambda, \xi)$ and D_{ik} are determined by formulae (35) and (40′). On the Earth surface $\xi = l$, taking into account $F_{00}F_{11} - F_{01}F_{10} = 1$, the above indicated formulae are transformed to the form

$$V_{1m}(l) = \frac{x_1}{D_{11}} \frac{1}{\sqrt{n_m}}$$

$$\frac{dV_{1m}}{d\xi}(l) = -\frac{x_1}{D_{11}} t_1 \frac{1}{\sqrt{n_m}} \quad (63)$$

$$V_{2m}(l) = \frac{x_2}{D_{10}} t_2 \frac{1}{\sqrt{n_m}}$$

$$\frac{dV_{2m}}{d\xi}(l) = -\frac{x_2}{D_{10}} \frac{1}{\sqrt{n_m}} \quad (64)$$

where $t_{1,2}$ are determined in section 4. At an effective height $\xi = 0$, we have

$$V_{1m}(0) = \frac{x_1}{D_{11}} D_{01} \frac{1}{\sqrt{n_m}}$$

$$\frac{dV_{1m}}{d\xi}(0) = x_1 \frac{1}{\sqrt{n_m}} \quad (65)$$

$$V_{2m}(0) = \frac{x_2}{D_{10}} D_{00} \frac{1}{\sqrt{n_m}}$$

$$\frac{dV_{2m}}{d\xi}(0) = x_2 \frac{1}{\sqrt{n_m}} \quad (66)$$

In order to obtain the functions of the conjugated operator $\mathbf{V}^{+*}$ and $d\mathbf{V}^{+*}/d\xi$, one should substitute x_1 to x_1^+ and x_2 to x_2^+ in formulae (59)–(66). Equation (41) we rewrite in the form

$$x_1(D_{01} - D_{11}\tau_{11}) = D_{11}\tau_{12}x_2$$

$$x_1 D_{10}\tau_{21} = (D_{00} - D_{10}\tau_{22})x_2$$

and from this obtain a characteristic equation

$$(D_{01} - D_{11}\tau_{11})(D_{00} - D_{10}\tau_{22}) = D_{11}D_{10}\tau_{12}\tau_{21}$$

If for the "m" mode (λ_m enters as a parameter into D_{ik})

$$|D_{10}\tau_{21}/(D_{00} - D_{10}\tau_{22})| < |D_{11}\tau_{12}/(D_{01} - D_{11}\tau_{11})|$$

we take

$$x_1 = 1 \qquad x_1^+ = 1$$

If

$$|D_{00} - D_{10}\tau_{22}| > |D_{11}\tau_{12}|$$

then

$$x_2 = \frac{D_{10}\tau_{21}}{D_{00} - D_{10}\tau_{22}} \qquad x_2^+ = \frac{D_{10}\tau_{12}}{D_{00} - D_{10}\tau_{22}} \quad (67)$$

In the opposite case

$$x_2 = \frac{D_{01} - D_{11}\tau_{11}}{D_{11}\tau_{12}} \qquad x_2^+ = \frac{D_{01} - D_{11}\tau_{11}}{D_{11}\tau_{21}}$$

If for the "m" mode

$$|D_{10}\tau_{21}/(D_{00} - D_{10}\tau_{22})| > |D_{11}\tau_{12}/(D_{01} - D_{11}\tau_{11})|$$

we take

$$x_2 = 1 \qquad x_2^+ = 1$$

if

$$|D_{01} - D_{11}\tau_{11}| > |\tau_{21}D_{10}|$$

then

$$x_1 = \frac{D_{11}\tau_{12}}{D_{01} - D_{11}\tau_{11}} \qquad x_1^+ = \frac{D_{11}\tau_{21}}{D_{01} - D_{11}\tau_{11}} \tag{68}$$

In the opposite case

$$x_1 = \frac{D_{00} - D_{10}\tau_{22}}{D_{10}\tau_{21}} \qquad x_1^+ = \frac{D_{00} - D_{10}\tau_{22}}{D_{10}\tau_{12}}$$

[34] According to (1), (3) and (9) we write the field in a regular waveguide in terms of Ψ_m

$$\tilde{\mathbf{\Phi}}(r, \Theta) = \frac{1}{\sqrt{\sin\Theta}} \sum \Lambda_m \mathbf{\Psi}_m(r) e^{i\nu_m\Theta} \tag{69}$$

Comparing (69) to formulae (46)–(51), we obtain the relation between $\tilde{\Lambda}_m$ and Λ_m

$$\Lambda_m = \frac{1}{2}\tilde{\Lambda}_m e^{+i\pi/4}\sqrt{a_m}/\sqrt{2\pi\nu_m} \tag{70}$$

We will name Λ_m a modified excitation coefficient.

[35] To find the excitation coefficients of modes by the antenna located at a height $(b - a)$ over the Earth surface and oriented in an arbitrary way we use the generalized reciprocity theorem for anisotropic media [*Felsen et al.*, 1973]. It follows from this theorem

$$\tilde{\mathcal{E}}_1 \mathbf{p}_2 = \tilde{\mathcal{E}}_2 \mathbf{p}_1 \tag{71}$$

where $\mathbf{p}_k = \int \mathbf{j}_k^e dV$ $(k = 1, 2)$, and in the case of short linear antenna considered here

$$\mathbf{p}_k = Jl_p \mathbf{l}_k$$

where J is the current at the antenna base, l_p is the antenna virtual height, $\mathbf{l}_k$ is a unit vector directed along the antenna, and $\tilde{\mathcal{E}}_1$ is the field of the antenna with the moment $\mathbf{p}_1$ in the waveguide filled by the medium with the dielectric permittivity $\hat{\varepsilon}(z)$, in the observation point coinciding to the position of the auxiliary source $\mathbf{p}_2$. On the Earth surface, the impedance δ_e is given and $\mathbf{p}_1$ is oriented in an arbitrary way. $\tilde{\mathcal{E}}_2$ is the field of the source $\mathbf{p}_2$ of a vertical short antenna in the point coincided with the source position $\mathbf{p}_1$. The waveguide is filled by the medium with a transposed tensor of the dielectric permittivity $\hat{\varepsilon}^T(z)$ and the same impedance on the Earth.

[36] Let the current momenta of both sources $|\mathbf{p}_1| = |\mathbf{p}_2|$ and heights $(b - a)$ of their position over the Earth surface coincide by magnitude. We present the field in the form of a sum

$$\mathcal{E}_1 = \sum \Lambda_m \mathbf{E}_m^{(1)} e^{i\nu_m\Theta} \tag{72}$$

over normal waves of the lateral operator K, (formula (2)). One has to determine Λ_m.

[37] In the second problem we present in the same way

$$\mathcal{E}_2 = \sum \Lambda_{m(2)} \mathbf{E}_m^{(2)} e^{i\nu_m\Theta} \tag{73}$$

$\mathbf{E}_m^{(2)}$ being the eigenfunctions of the operator $K^{(2)}$ which is different from K^{+*} (formula (5′)) in the following way: (1) the sign at ∇_t is changed and (2) the sign in the boundary conditions is changed. These differences are compensated by the changes in the sign of $\mathbf{l}_\Theta$ in the second problem. The eigenvalues ν_m in (72) and (73) coincide. Formulating the problem, we have noted that the receiver and transmitter are located in the near-ground layer of the atmosphere below the ionosphere, therefore

$$E_{m\Theta}^{(2)} = -E_{m\Theta}^{(1)+*} \qquad H_{m\Theta}^{(2)} = -H_{m\Theta}^{(1)+*}$$

$$E_{mr}^{(2)} = E_{mr}^{(1)+*} \qquad E_{m\varphi}^{(2)} = E_{m\varphi}^{(1)+*}$$

$$H_{mr}^{(2)} = H_{mr}^{(1)+*} \qquad H_{m\varphi}^{(2)} = H_{m\varphi}^{(1)+*}$$

then

$$E_{mr}^{(2)+*} = E_{mr}^{(1)}$$

Let

$$\mathbf{E}_m = \mathbf{E}_m^{(1)}$$

$$\Lambda_{m(2)} = -Jl_p\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{k_0 r_{\mathrm{ef}}}{2Mb}V_{1m}(\xi_b)\sqrt{a_m}/\sqrt{2\pi\nu_m}$$

Using equality (71) we obtain

$$\sum_m \left[Jl_p \cdot \Lambda_m E_{mr}^{(1)} - \Lambda_{m(2)}(\mathbf{E}_m^{(2)}\mathbf{p}_1)\right] e^{i\nu_m\Theta} = 0 \tag{74}$$

or, equalizing to zero each term in the sum, we obtain

$$\Lambda_m = \Big\{\Lambda_{m(2)}(E_{mr}^{+*}(\xi_b)\cos\Theta_a -$$

$$E_{m\Theta}^{+*}(\xi_b)\sin\Theta_a\cos\varphi_a +$$

$$E_{m\varphi}^{+*}(\xi_b)\sin\Theta_a\sin\varphi_a)\Big\}\Big/E_{mr}(\xi_b)$$

the following formula being valid

$$\Lambda_m = -\frac{120\pi\cdot Jl_pk_0r_{\rm ef}}{2Mb\sqrt{2\pi\nu_m}}\sqrt{a_m}\Big[V_1^{+*}(\xi_b)\cos\Theta_a +$$

$$\frac{i}{\nu_m}\left(\frac{1}{2}V_1^{+*}(\xi_b) - 2M^2\frac{dV_1^{+*}(\xi_b)}{d\xi}\right)\sin\Theta_a\cos\varphi_a +$$

$$k_0b/\nu_mV_2^{+*}(\xi_b)\sin\Theta_a\sin\varphi_a\Big] \quad (75)$$

$$\xi_b = \frac{k_0r_{\rm ef}}{M}ln\frac{r_{\rm ef}}{b}$$

[38] The modified excitation coefficients Λ_m are presented in millivolts, if the current in the antenna J, antenna length l_p, and source coordinate b are expressed in amperes, meters, and kilometers, respectively.

[39] In the real conditions, the Earth-ionosphere waveguide appears irregular because of the inhomogeneity of geophysical conditions (conductivity of the Earth surface, illumination of the path, and magnetic field of the Earth). In the model we use, the real waveguide is presented as a piecewise-homogeneous one.

[40] At the homogeneous piece with number 0 in the vicinity of the transmitter we find the eigenvalues $\nu_m^{(o)}$ and eigenfunctions $\mathbf{\Psi}_m^{(o)}$, and take $\Lambda_m^{(o)} = \Lambda_m$. Using formula (12) we determine the matrix of transformation of normal waves $\Pi_{nm}^{(1)} = (1/2)(\mathbf{\Psi}_n^{(1)+}, \Gamma\mathbf{\Psi}_m^{(o)})$ at the joint boundary of homogeneous pieces, and then calculate the amplitudes of the normal waves falling onto the boundary of the next to number N homogeneous piece by the formula

$$\Lambda_j^{(N)} = \sum_i \Pi_{ji}^{(N)}\Lambda_i^{(N-1)}e^{i\nu_i^{(N-1)}\Delta\Theta^{(N-1)}} \quad (76)$$

$$\Pi_{nm}^{(N)} = \frac{1}{2}(\mathbf{\Psi}_n^{(N)+}, \Gamma\mathbf{\Psi}_m^{(N-1)})$$

[41] The field in the irregular waveguide we find using the formula

$$\tilde{\mathbf{\Phi}}(r,\Theta) = \frac{1}{\sqrt{\sin\Theta}}\sum_m \mathbf{\Psi}_m^{(N)}(r)\Lambda_m^{(N)}e^{i\nu_m^{(N)}(\Theta-\Theta^{(N)})} \quad (77)$$

[42] Thus for calculation of any component of the field at any height in the vacuum cavity of the irregular waveguide using formula (77), we have formulae for $\Lambda_m^{(0)}$; $\Lambda_m^{(N)}$ is calculated using formula (76) with the help of the transformation matrix $\Pi_{nm}^{(N)}$. Elements of the latter matrix may be calculated approximately by the formula

$$\Pi_{nm} = \frac{1}{\lambda_n^{(2)} - \lambda_m^{(1)}}\left[V_n^{(2)+*}\frac{dV_m^{(1)}}{d\xi} - \right.$$

$$\left.V_m^{(1)}\frac{dV_n^{(2)+*}}{d\xi}\right]_0^l\cdot\left(\frac{a}{r_{\rm ef}}\right)^{4/3}$$

The eigenfunctions (52)–(57) are recalculated to any height by the formulae (59)–(62). The approximated value of the normalized factor is calculated by formula (58).

[43] The relative error Δ of calculation of the normalizing factor is estimated by the formula

$$\Delta = \frac{\int\limits_{r_{\rm ef}}^{\infty}\mathbf{\Psi}_m^{+*}\Gamma\mathbf{\Psi}_m rdr}{\int\limits_{a}^{r_{\rm ef}}\mathbf{\Psi}_m^{+*}\Gamma\mathbf{\Psi}_m rdr} =$$

$$(x_1^+(\dot{\tau}_{11}x_1 + x_2\dot{\tau}_{12}) + x_2^+(\dot{\tau}_{21}x_1 + x_2\dot{\tau}_{22}))\frac{1}{M^2n_m} \quad (78)$$

for every mode m.

[44] If the observation point is located below the surface level, the field first is calculated on the Earth surface by the above described formulae $\tilde{\mathcal{E}}(a)$, and then the components $\tilde{\mathcal{E}}_\Theta$ and $\tilde{\mathcal{E}}_\varphi$ are multiplied to $\exp(-ik_Ed_a)$, where $d_a < 0$ is the depth of the receiver location, $k_E = k_0\sqrt{\varepsilon_E}$, $\varepsilon_E = \varepsilon + (i\sigma/\omega\varepsilon_0)$, ε is the relative dielectric permittivity, and σ is the conductivity of the medium where the receiver is located,

$$\tilde{\mathcal{E}}_{\Theta,\varphi}(a+d_a,\Theta) = \tilde{\mathcal{E}}_{\Theta,\varphi}(a,\Theta)\times\exp(-ik_Ed_a) \quad (79)$$

$$\tilde{\mathcal{E}}_r(a+d_a,\Theta) = \tilde{\mathcal{E}}_r(a,\Theta)\exp(-ik_Ed_a)/\varepsilon_E \quad (80)$$

We do the same for three components of the magnetic field

$$\tilde{\mathcal{H}}_{r,\Theta,\varphi}(a+d_a,\Theta) = \tilde{\mathcal{H}}_{r,\Theta,\varphi}(a,\Theta)\times\exp(-ik_Ed_a) \quad (81)$$

The phase of the components of the field $\tilde{\Phi}$ is determined relative to the phase of the current J at the antenna input.

6. Conclusion

[45] This paper is mainly of a review character and compiles the existing methods of description of the field in a spherical anisotropic waveguide channel Earth-ionosphere with the dependence of the medium properties on two spatial coordinates at its excitation by an arbitrary oriented short antenna. A two-dimensionally inhomogeneous waveguide is modeled by a sequence of homogeneous pieces. To accelerate the iteration process while looking for the eigenvalues, it is proposed to use an effective height of the waveguide channel calculated as a height below which vacuum is located at the least dependence of the elements of the matrix of the reflective properties of the ionosphere on the spectral parameter for the main mode. It is shown that the use of

a bivector at calculation of the reflective characteristics of the ionosphere makes it possible to avoid an appearance of a singularity. For acceleration of the calculations it is proposed to calculate the coefficients of reexcitation of normal waves over the vacuum region, the control of the reexcitation matrix elements accuracy being performed by the above described method.

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