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# Propagation of electromagnetic waves in a longitudinally inhomogeneous waveguide at the presence of singular points

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[1] The problem of wave propagation in a two-dimensional waveguide with vertical absolutely thin barrier and ideally conducting walls is solved in the paper. This is a model approximating the radio wave propagation in the Earth–ionosphere waveguide. The main attention in the model is drawn to the influence of irregularities of the Earth surface including the influence of singular points of an angular type. The presence of the smooth irregularity of the upper wall, i.e., ionosphere, is also taken into account. Two methods of the solution are used: the method of semi-inversion developed by the authors and the method of quasi-static Green function. Using these methods, the diffraction of the field of a dipole at the vertical barrier in a waveguide with curved upper wall and in a waveguide with plain upper wall is calculated. On the basis of the obtained solutions a conclusion is drawn on the influence of ionospheric disturbances on the field. *INDEX TERMS:* 2487 Ionosphere: Wave propagation; 2439 Ionosphere: Ionospheric irregularities; 2411 Ionosphere: Electric fields; *KEYWORDS:* VLF waves propagation; Ionospheric waveguide; Ionospheric irregularities.

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## 1. Introduction

[2] In solving of many practical problems concerning electromagnetic field propagation one has to deal with cases of the presence of singular points (sharp edge, corners, etc.) at the boundary of the propagation region. Modeling of the problem of radio wave propagation in the Earth–ionosphere waveguide is one of such problems. One of the main aims in this problem is to take into account irregularities which may exist at the lower wall (the Earth) and especially of angular points (mountain ridges, electric power lines) which lead to appearance of peculiarities in the fields. The ionospheric disturbances are smooth or absent. As far as the main attention is paid to the role of irregularities, the walls of the model waveguide are chosen to be conducting ideally.

[3] The presence of angular points at the boundary of the propagation region even in the simplest cases complicates considerably numerical solution of the problem: the equation systems obtained by the simple joining method have unlimited in $l_2$ operators, whereas the truncation method has in the best case a conventional (i.e., dependent of the truncation method) convergence [*Mittra and Li*, 1974]. So solving such problems one should use some regularization methods. The main methods of solving such problems are: the method of inversion of the residuals of part of the matrix operator [*Shestopalov et al.*, 1973] the method of residues [*Mittra and Li*, 1974], the method of quasi-static Green function (MQGF) [*Verbitsky*, 1981], and also the semi-inversion method (SIM) [*Maison and Makarov*, 1996], the latter two methods being used in this paper.


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## 2. Formulation of the Problem

[4] In this paper the problem of diffraction of electromagnetic waves in the Earth–ionosphere waveguide is solved, the waveguide being modeled by a plain waveguide with a vertical barrier. The walls and barrier are infinitely conducting.

[5] In the beginning, the problem of diffraction in a plain waveguide with a thin wall and smoothly curved upper wall (that corresponds to the presence of a smooth disturbance in the ionosphere) is solved with the help of SIM. Taking into account some introduced limitation of the problem parameters, the zero and first approximations are found. Then a two-dimensional waveguide with a barrier and plain upper

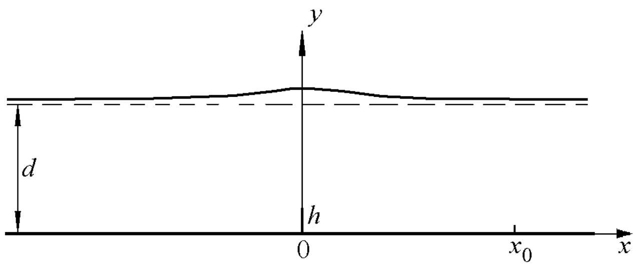


**Figure 1.** Geometry of the problem.

wall (undisturbed ionosphere) is considered. The problem is solved using MQGF. This makes it possible to get rid of some limitations having appeared at the use of SIM. The comparison of the results of application of these methods makes it possible to conclude that the input of smooth disturbances of the ionosphere into the electromagnetic field is insignificant as compared to the influence of local angular irregularities on the Earth surface.

[6] Now we come to consideration of a two-dimensional waveguide with a curved upper wall. A vertical barrier with a height $h$ is located in the point $(0,0)$. The upper wall of the waveguide is described by

$$y = d\sqrt{1 + \frac{h^2}{d^2 + x^2}}$$

A thread of vertical dipoles with the dipole momentum density $\mathbf{j} = -\mathbf{e}_y A_0 \delta(x - x_0)\delta(y)$ is a source of the field. The thread is located at the lower wall in the point $(x_0, 0)$, the irregularity being located in the remote zone of the source: $x_0 \gg d$ (Figure 1).

## 3. Solution and Results

[7] It is easy to see that the Maxwell equations are reduced to the Helmholtz equations for $H_z$ of the following form:

$$\nabla^2 H_z + k^2 H_z = A_0 \delta'(x - x_0)\delta(y)$$

with the boundary condition $\partial H_z/\partial n|_\Gamma$, where $\Gamma$ is the outline of the region and $n$ is a normal to it.

[8] The point $(0, h)$ is a singular point of the Helmholtz operator. In order to get rid of the singularity we perform a conform transformation of coordinates transforming the lower and upper walls of the waveguide into planes. It is easy to see that the transformation $\omega = \sqrt{t^2 + h^2}$ (where $t = x + iy$ are the physical coordinates, $\omega = \xi + i\eta$ are the conform coordinates, and the branch of the root is taken in such a way that $\mathrm{Im}\sqrt{t^2 + h^2} > 0$) satisfies this requirement.

[9] It follows from the condition $x_0 \gg d$ that $\xi_0 \gg d$. So we find

$$\frac{\partial}{\partial x}(\delta(\xi - \xi_0)\delta(\eta)) \approx \frac{1}{h_\xi^2}\delta'(\xi - \xi_0)\delta(\eta)$$

and then the initial equation takes the form

$$\nabla^2 H_z + k^2 h_\xi^2 H_z = A_0 \delta'(\xi - \xi_0)\delta(\eta) \quad (1)$$

Here

$$h_\xi = h_\eta = \left|\frac{dt}{d\omega}\right| = \frac{|\omega|}{\sqrt{|\omega^2 - h^2|}}$$

is the Lame coefficient of the performed transformation. Inverting (1) by the Kirchoff's formula, we obtain the integral equation

$$H_z = H_z^{(0)} - k^2 \int_{-\infty}^{+\infty}\int_0^d G(\xi,\eta;\xi',\eta')(1 - h_\xi^2(\xi',\eta'))\times$$

$$H_z(\xi',\eta') d\eta d\xi \quad (2)$$

where $G(\xi,\eta;\xi',\eta')$ is the Green function for the Neumann problem and

$$H_z^{(0)} = -A_0 \int_{-\infty}^{+\infty}\int_0^d G(\xi,\eta;\xi',\eta')\delta'(\xi' - \xi_0)\delta(\eta') d\eta' d\xi' =$$

$$-\frac{A_0}{2d}\,\mathrm{sign}(\xi_0 - \xi)\sum_{n=0}^{\infty} e^{i\lambda_n|\xi_0 - \xi|}\cos(\frac{\pi n}{d}\eta) \quad (3)$$

The solution for the vacuum having a correct behavior at the infinity is taken as $H_z^{(0)}$. Under this choice and taking into account that $h_\xi \to 1$ if $|\xi| \to \infty$, the limitation of the integral operator in $L_2$ is provided.

[10] We will solve equation (2) by the sequential approximations method. For its application one has to be sure that the norm of the integral operator is less than unity:

$$K\Psi = k^2 \int_{-\infty}^{+\infty}\int_0^d G(\xi,\eta;\xi',\eta')(1 - h_\xi^2(\xi',\eta'))\times$$

$$\Psi(\xi',\eta') d\eta' d\xi'$$

Using asymptotics for the Lame coefficient, one can show that under the condition $k < \pi/d$ (a low-frequency case, i.e., a one-mode waveguide) the operator norm has the form:

$$\|K\| \sim kh\frac{h}{d}$$

Therefore, if the conditions $kh(h/d) < 1$ and $k < \pi/d$ are fulfilled, one can apply the sequential approximations method. We take $H_z^{(0)}$ (3) as a zero approximation. Then we find the first approximation using

$$H_z^{(1)} = H_z^{(0)} - k^2 \int_{-\infty}^{+\infty}\int_0^d G(\xi,\eta;\xi',\eta')(1 - h_\xi^2(\xi',\eta'))\times$$

$$H_z^{(0)}(\xi',\eta') d\eta' d\xi' = H_z^{(0)} + \Delta H^{(1)}$$

where

$$\Delta H^{(1)} = -k^2 \int_{-\infty}^{+\infty}\int_0^d (-\frac{1}{2id})\sum_{q=0}^{\infty}\frac{1}{\lambda_q}e^{i\lambda_q|\xi-\xi'|}\times$$

$$\cos(\frac{\pi q}{d}\eta)\cos(\frac{\pi q}{d}\eta')(1-h_\xi^2(\xi',\eta'))(-\frac{A_0}{2d})\times$$

$$\sum_{n=0}^{\infty}e^{i\lambda_n(\xi_0-\xi')}\cos(\frac{\pi n}{d}\eta')d\eta' d\xi' =$$

$$\frac{ik^2A_0}{4d^2}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}\frac{e^{i\lambda_n\xi_0}}{\lambda_q}[e^{i\lambda_q\xi}A_{nq}+e^{-i\lambda_q\xi}B_{nq}\cos(\frac{\pi q}{d}\eta)$$

Here $A_{nq}$ and $B_{nq}$ are the coefficients of propagation and reflection of the $q$th wave caused by its interaction with the $n$th wave.

[11] Since all waves (except the zero one) are local waves, $\xi_0 \gg d$, and $|\xi| \gg d$, the only significant input into the field is provided by the $A_{00}$ and $B_{00}$ coefficients. Therefore the further solution is reduced to finding these coefficients.

[12] Neglecting by the terms of infinitesimal of higher orders of the $kh(h/d)$ parameters we obtain

$$A_{00}(\xi) \approx -\frac{idh^2}{2k\xi^2}e^{-2ik\xi}$$

$$B_{00} \approx -0.79h^2 - \frac{dh^2}{\xi}$$

under $\xi < -d$ and

$$A_{00} \approx 2.4h^2 - i\frac{h^2}{2}\frac{d}{\xi}\frac{e^{-2ik\xi}}{k\xi}$$

$$B_{00} \approx -h^2\frac{d}{\xi}$$

under $\xi > d$.

[13] Since all local waves are exponentially small (because $\xi_0 \gg d$ and $|\xi| > d$), one can write an expression for the field taking into account only the zero modes. Taking into account that the norm $K$ has an order of $kh(h/d)$, one has in the first approximation to exclude all the terms of the higher infinitesimal degree. This leads to a fact that in the first approximation one can obtain a correction to the zero approximation only in phase.

[14] To the left from the barrier (that is in the opposite relative to the source part of the waveguide) the expression for the field has the form

$$H_z^{(1)}(\xi,\eta) \approx -\frac{A_0e^{ik\xi_0}}{2d}\times$$

$$\left[e^{-ik\xi}+0.39ikh\frac{h}{d}e^{-ik\xi}+\frac{i}{2}kh\frac{h}{\xi}e^{-ik\xi}\right] \quad (4)$$

whereas to the right from the barrier it is written as

$$H_z^{(1)}(\xi,\eta) \approx -\frac{A_0e^{ik\xi_0}}{2d}\times$$

$$\left[e^{-ik\xi}+\frac{i}{2}kh\frac{h}{\xi}e^{-ik\xi}-1.2ikh\frac{h}{d}e^{ik\xi}\right] \quad (5)$$

The electric field at the barrier in the zero approximation is written in the following way:

$$E_\eta = \frac{kA_0}{2d\omega\varepsilon_0}\frac{|y|}{\sqrt{|y^2-h^2|}}e^{ikx_0}e^{-ik\sqrt{h^2-y^2}}$$

The behavior of the field in the vicinity of a singular point is mainly determined by the Lame coefficient. In the zero approximation in the vicinity of the singular point ($\xi = 0,\ \eta = 0$)

$$|E_\eta| \sim \frac{1}{\sqrt{h^2-y^2}} \sim \frac{1}{\sqrt{r}}$$

which corresponds to the Meixner condition for a semiplane.

[15] Transferring (4) and (5) into physical coordinates and neglecting the terms of higher orders by $kh(h/d)$ we obtain

$$H_z^{(1)}(x,y) \approx -\frac{A_0e^{ikx_0}}{2d}\left[e^{-ikx}+0.39ikh\frac{h}{d}e^{-ikx}\right] \quad (4a)$$

under $x < -d$ and

$$H_z^{(1)}(x,y) \approx -\frac{A_0e^{ikx_0}}{2d}\left[e^{-ikx}-1.2ikh\frac{h}{d}e^{ikx}\right] \quad (5a)$$

under $x > d$.

[16] Taking into account that $kh(h/d) \ll 1$ one can rewrite (4a) and (5a) in the form

$$H_z^{(1)}(x,y) \approx -\frac{A_0e^{ikx_0}}{2d}e^{-ikx}e^{i\Phi_1} \qquad \Phi_1 = 0.39kh\frac{h}{d}$$

under $x < -d$ and

$$H_z^{(1)}(x,y) \approx -\frac{A_0e^{ikx_0}}{2d}e^{-ikx}e^{i\Phi_2} \qquad \Phi_2 = 1.2kh\frac{h}{d}$$

under $x > d$, where $\Phi_{1,2}$ are the corrections to the phase to the left and to the right from the barrier, respectively.

[17] Thus the obtained solution does not contain terms with the propagation and reflection coefficients depending on $x$. Both, the reflected and falling waves have constant amplitudes (at least with the accuracy to the second order of infinitesimal) also at $|x| > d$. Therefore the local irregularity (i. e., the barrier) provides the main influence on the propagation, but not the distributed one (curved upper wall).

[18] Now we come to a consideration of the problem of diffraction in a similar waveguide but with a plain upper wall (Figure 2). Let us assume that from the left a normal wave falls on the barrier. The wave is a zero one for the TM field ($U_0 = e^{ikx}$) and a first one for the TE field ($U_0 = e^{i\lambda_1 x}\sin\pi y/d$, where $U$ corresponds to $H_z$ and $E_z$

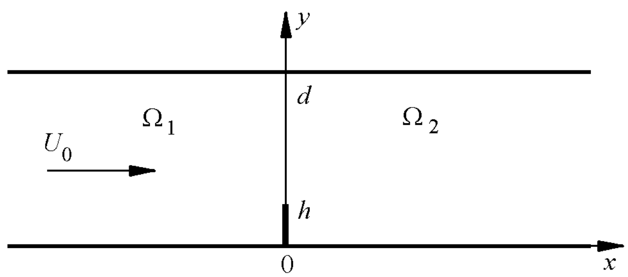


**Figure 2.** Scheme of the waveguide with a plain upper wall.

for the transversal magnetic (TM) and transversal electric (TE) fields, respectively). This is equivalent to consideration of the field of a dipole located far from the obstacle in a one-mode waveguide. According to the general scheme of MQGF we present the field in the waveguide as a sum of normal waves:

[19] TE case:
In the $\Omega_1$ region

$$u_1 = e^{i\lambda_1 x} \sin \frac{\pi y}{d} + \sum_{n=1}^{\infty} R_n e^{-i\lambda_n x} \sin \frac{\pi n}{d} y$$

In the $\Omega_2$ region

$$u_2 = \sum_{n=1}^{\infty} I_n e^{i\lambda_n x} \sin \frac{\pi n}{d} y \tag{6}$$

TM case:
In the $\Omega_1$ region

$$u_1 = e^{ikx} + \sum_{n=0}^{\infty} R_n e^{-i\lambda_n x} \cos \frac{\pi n}{d} y$$

in the $\Omega_2$ region

$$u_2 = \sum_{n=0}^{\infty} I_n e^{i\lambda_n x} \cos \frac{\pi n}{d} y \tag{7}$$

Then at the boundary of the regions we will join solutions assuming equality of the fields and their normal derivatives and also their being equal to zero at the barrier.

[20] According to the general scheme of MQGF, the Green function of the Laplace operator (satisfying the same boundary conditions as the solution) is used for the regularization of the system (6) or (7). Multiplying the fields and their derivatives by the normal derivative of the Green function and by the function itself, respectively, we integrate the fields over the boundary between regions. It can be easily shown [*Konorov and Makarov*, 1987] that such procedure is equivalent to a conversion of the Laplace operator.

[21] The functional equations obtained by the joining are expanded in a series about some complete system of functions and lead to a system of linear algebraic equations. According to the general MQGF theory [*Konorov and Makarov*, 1987; *Verbitsky*, 1981] the system is a regular one allowing for a truncation. Its matrix elements are double integrals of the Green function and functions used in the expansion.

[22] To find the Green function, we use a conformal transformation of coordinates

$$\omega(z) = \frac{2d}{\pi} \operatorname{arth} \left[ \cos \frac{\pi h}{2d} \sqrt{\tanh^2 \frac{\pi z}{2d} + \tan^2 \frac{\pi h}{2d}} \right]$$

converting the initial waveguide into a plain one. The branch of the root is fixed in the following way:

$$\operatorname{Im} \sqrt{\tanh^2 \frac{\pi z}{2d} + \tan^2 \frac{\pi h}{2d}} > 0 \quad (z = x + iy,\, \omega = \xi + i\eta)$$

Then the Green function for the Dirichlet's and Neumann problems takes, respectively, the form

$$G_D(x, y; x', y') = -\frac{1}{\pi} \sum_{q=1}^{\infty} \frac{1}{q} e^{-\frac{\pi q}{d} |\xi(x,y) - \xi(x',y')|} \times$$

$$\sin \frac{\pi q}{d} \eta(x,y) \sin \frac{\pi q}{d} \eta(x',y')$$

and

$$G_N(x, y; x', y') = -\frac{1}{d} \left[ \begin{array}{ll} \xi(x,y) & \text{at } \xi(x,y) > \xi(x',y') \\ \xi(x',y') & \text{at } \xi(x',y') > \xi(x,y) \end{array} \right.$$

$$-\frac{1}{\pi} \sum_{q=1}^{\infty} \frac{1}{q} e^{-\frac{\pi q}{d} |\xi(x,y) - \xi(x',y')|} \times$$

$$\cos \frac{\pi q}{d} \eta(x,y) \cos \frac{\pi q}{d} \eta(x',y')$$

We consider first the Dirichlet's problem. Following the general scheme we obtain the functional equation

$$\int_0^d \left( \frac{\partial G}{\partial x} - i\lambda_1 G \right) \Big|_{x=-0} \sin \frac{\pi y}{d} dy -$$

$$\sum_{n=1}^{\infty} R_n \int_0^d \left( \frac{\partial G}{\partial x} + i\lambda_n G \right) \Big|_{x=-0} \sin \frac{\pi n}{d} y dy -$$

$$\sum_{n=1}^{\infty} I_n \int_0^d \left( \frac{\partial G}{\partial x} - i\lambda_n G \right) \Big|_{x=+0} \sin \frac{\pi n}{d} y dy = 0 \tag{8}$$

Applying the second Green formula to the Green function and function in the and regions we find

$$\int_0^d \frac{\partial G}{\partial x} \Big|_{x=-0} \sin \frac{\pi n}{d} y dy =$$

$$X'(\Omega_1)e^{\frac{\pi n}{d}x}\sin\frac{\pi n}{d}y+\frac{\pi n}{d}\int_0^d G\Big|_{x=0}\sin\frac{\pi n}{d}y$$

$$\int_0^d \frac{\partial G}{\partial x}\Big|_{x=-0}\sin\frac{\pi n}{d}ydy=$$

$$-X'(\Omega_2)e^{-\frac{\pi n}{d}x}\sin\frac{\pi n}{d}y-\pi nd\int_0^d G\Big|_{x=0}\sin\frac{\pi n}{d}y \qquad (9)$$

where

$$X'(\Omega)=\left[\begin{array}{ll}1 & (x',y')\in\Omega\\ 0 & (x',y')\notin\Omega\end{array}\right.$$

Using (9), we get rid of the derivative of the Green function in (8):

$$-X'(\Omega_1)e^{-\frac{\pi}{d}x'}\sin\frac{\pi}{d}y'+(\frac{\pi}{d}-i\lambda_1)\int_0^d G|_{x=0}\sin\frac{\pi n}{d}y+$$

$$\sum_{n=1}^{\infty}R_n\left[X'(\Omega_1)e^{-\frac{n\pi}{d}x'}\sin\frac{\pi n}{d}y'+\right.$$

$$\left.(\frac{\pi n}{d}+i\lambda_n)\int_0^d G|_{x=0}\sin\frac{\pi n}{d}y\right]+$$

$$\sum_{n=1}^{\infty}I_n\left[X'(\Omega_2)e^{-\frac{n\pi}{d}x'}\sin\frac{\pi n}{d}y'+\right.$$

$$\left.(\frac{\pi n}{d}+i\lambda_n)\int_0^d G|_{x=0}\sin\frac{\pi n}{d}y\right]=0 \qquad (10)$$

Equation (10) at different $x'$ is separated into two equations:

[23] In the $\Omega_1$ region

$$e^{\frac{\pi}{d}x'}\sin\frac{\pi}{d}y'+(\pi d-i\lambda_1)\int_0^d G\Big|_{x=0}\sin\frac{\pi y}{d}dy+$$

$$\sum_{n=1}^{\infty}R_n\left[e^{\frac{n\pi}{d}x'}\sin\frac{\pi n}{d}y'+\ \rho_n\int_0^d G|_{x=0}\sin\frac{\pi n}{d}ydy\right]+$$

$$\sum_{n=1}^{\infty}I_n\rho_n\int_0^d G|_{x=0}\sin\frac{\pi n}{d}ydy=0 \qquad (11)$$

In the $\Omega_2$ region

$$(\pi d-i\lambda_1)\int_0^d G\Big|_{x=0}\sin\frac{\pi y}{d}dy+$$

$$\sum_{n=1}^{\infty}R_n\rho_n\int_0^d G|_{x=0}\sin\frac{\pi n}{d}ydy+$$

$$\sum_{n=1}^{\infty}I_n\left[e^{-\frac{n\pi}{d}x'}\sin\frac{\pi n}{d}y'+\right.$$

$$\left.\rho_n\int_0^d G|_{x=0}\sin\frac{\pi n}{d}ydy\right]=0 \qquad (12)$$

where the denotation $\rho_n=(\pi n/d)+i\lambda_n$ is introduced.

[24] Expressions (11) and (12) are functional equations. To move to linear algebraic system, one has to expand these equations in some complete system of functions. It is convenient to use $\{\sin(\pi p/d)y'\}$ as such functions. Multiplying (11) and (12) by $\sin(\pi p/d)y'$ and integrating with respect to $y'$ between 0 and $d$ we obtain

$$e^{\frac{\pi}{d}x'}\frac{d}{2}\delta_{1p}+(\pi d-i\lambda_1)\int_0^d\int_0^d G\Big|_{x=0}\times$$

$$\sin\frac{\pi y}{d}\sin\frac{\pi p}{d}y'dydy'+$$

$$\frac{d}{2}R_pe^{\frac{np}{d}x'}+\sum_{n=1}^{\infty}R_n\rho_n\int_0^d\int_0^d G\Big|_{x=0}\times$$

$$\sin\frac{\pi n}{d}y\sin\frac{\pi p}{d}y'dydy'+$$

$$\sum_{n=1}^{\infty}I_n\rho_n\int_0^d\int_0^d G\Big|_{x=0}\sin\frac{\pi n}{d}y\sin\frac{\pi p}{d}y'dydy'=0 \qquad (13)$$

and

$$(\pi d-i\lambda_1)\int_0^d\int_0^d G\Big|_{x=0}\times$$

$$\sin\frac{\pi y}{d}\sin\frac{\pi p}{d}y'dydy'+$$

$$\sum_{n=1}^{\infty}R_n\rho_n\int_0^d\int_0^d G\Big|_{x=0}\sin\frac{\pi n}{d}y\sin\frac{\pi p}{d}y'dydy'+$$

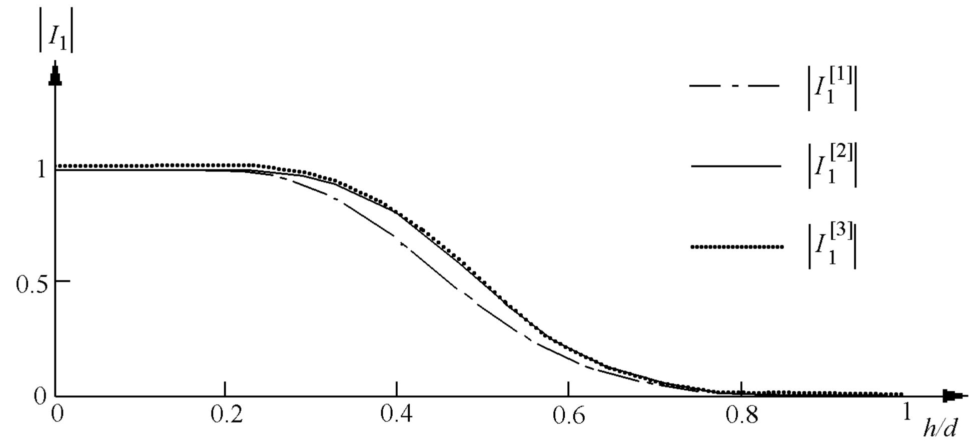


**Figure 3.** Behavior of the modulus of the propagation coefficient of the first mode as a function of the barrier height for $d = 10$ and $k = 0.5$.

$$\frac{d}{2} I_p e^{\frac{np}{d}x'} + \sum_{n=1}^{\infty} I_n \rho_n \int_0^d \int_0^d G\Big|_{x=0}$$

$$\sin \frac{\pi n}{d} y \sin \frac{\pi p}{d} y' dy dy' = 0 \qquad (14)$$

Taking into account that (13) and (14) should be fulfilled at any $x'$, we take $x' = 0$ and obtain the final system of equations

$$\left.\begin{array}{l} \sum_{n=1}^{\infty} I_n B_{np} + \dfrac{I_p}{2} = i\lambda_1 B_{1p} \\ I_p - R_p = \delta_{1p} \end{array}\right\} \qquad (15)$$

in which

$$B_{np} = \frac{2}{d} \int_0^d \int_0^d G\Big|_{x=0, x'=0} \sin \frac{\pi n}{d} y \sin \frac{\pi p}{d} y' dy dy' = 0 \qquad (16)$$

In a similar way we derive the system of equations in the case of the Neumann problem. Its principal difference from the Dirichlet's problem is that the Green function does not tend to zero at the infinity. So the system of functions $\{\cos(\pi p/d)y'\}_{p\neq 0}$ is not complete and one should add to it $[1; x']$: projecting to these functions would give "zero" equations:

$$\left.\begin{array}{rl} -ikB_{0p}^{--} + R_p + \sum_{n=0}^{\infty} \rho_n R_n B_{np}^{+-} & = 0 \\ -ikB_{0p}^{-+} + I_p + \sum_{n=0}^{\infty} \rho_n R_n B_{np}^{-+} & = 0 \\ p & \neq 0 \end{array}\right\}$$

$$\left.\begin{array}{rl} 2 + ikI_0(\beta_{02} - \beta_{01}) \times & \\ \sum_{n=1}^{\infty} \rho_n (R_n \beta_{n1} - I_n \beta_{n2}) & = 0 \\ I_0 + R_0 & = 1 \end{array}\right\} \qquad (17)$$

where

$$B_{mn}^{\pm\pm} = \frac{2}{d} \int_0^d \int_0^d G\Big|_{x=\pm 0, x'=\pm 0} \cos \frac{\pi n}{d} y \cos \frac{\pi m}{d} y' dy dy'$$

Thus regular systems of equations for the Dirichlet's and Neumann problems are obtained. Equations (15) and (17) allow for a reduction and (finding matrix elements) one can easily obtain the solution with the required accuracy, that is, in the numerical sense the problem may be considered as solved. However, if one is interested in analytical form of the solutions, one should study the $B_{mn}$ dependency on the number of the element and waveguide parameters.

[25] In analytical solving the MQGF problems the main difficulty is determination of the matrix elements of the equation system, the elements being double integrals of the Green functions (16). To obtain the analytical dependence of the matrix elements on the problem parameters, we use the method suggested by *Konorov and Makarov* [1987].

[26] To do this, one has to obtain the expression of the matrix integrals in terms of the expansion coefficients of a conformal transformation by exponents. By rather inconvenient transformations one can obtain an expression for $B_{mn}$ in the form of finite sums, the complexity of these sums growing quickly with an increase of the matrix element number.

[27] We will not describe in details the calculation of the matrix elements but present the first approximations (solutions of the first equation of the system only) for the Dirichlet's ($I_1^{(1)}$) and Neumann ($I_0^{(1)}$) problems:

$$I_1^{(1)} = \frac{i\lambda_1 B_{11}}{\rho_1 B_{11} + \dfrac{1}{2}} = \frac{id}{2\pi} \sqrt{k^2 - \left(\frac{\pi}{d}\right)^2} \times$$

$$\frac{\cos^4 \dfrac{\pi h}{2d}}{-\dfrac{d}{2\pi} \cos^4 \dfrac{\pi h}{2d} \left( \dfrac{\pi}{d} + i\sqrt{k^2 - \left(\dfrac{\pi}{d}\right)^2} \right)}$$

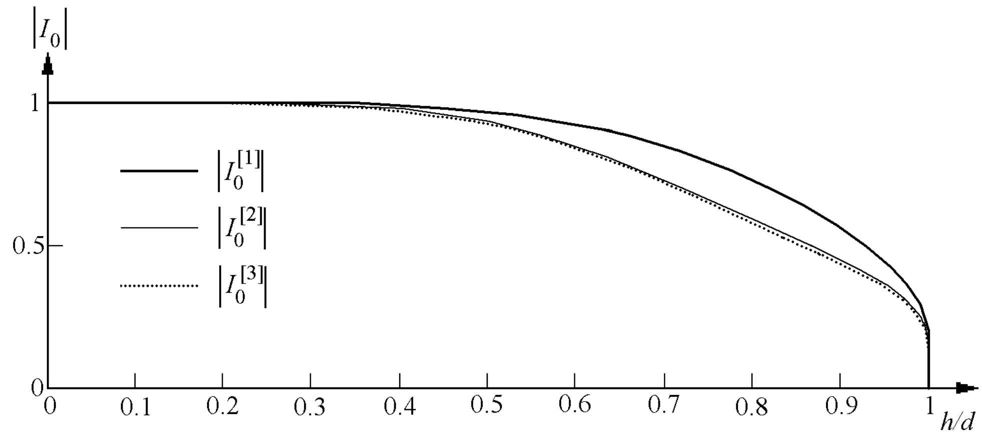


**Figure 4.** Same as Figure 3 but for $d = 10$, $k = 0.2$, and for TM waves.

$$I_0^{(1)} = \frac{1}{1 - ik\beta_{02}} = \frac{1}{1 + i\dfrac{kd}{\pi}\ln(\cos\dfrac{\pi h}{2d})} \quad (18)$$

It is worth noting that unlike the first method considered there are no limitations to the barrier height $h$ in this solution.

[28] Analytical expressions for the second and third approximations are also obtained; however, they are rather massive. Comparing the solutions of systems containing one, two, and three equations, one can study the correction introduced into the solution by every new equation and evaluate the accuracy of the final result.

[29] Analyzing the one-mode waveguide one can see that to find the solution at small and large $h$ it is enough to consider one equation. If $h \approx d/2$, to have more exact determination of $I_1$ one should take the system of two equations. The third equation introduces the correction not exceeding 10%. Figure 3 shows the behavior of the modulus of the propagation coefficient of the first mode as a function of the barrier height for $d = 10$ and $k = 0.5$.

[30] The behavior of the corrections in the first and second approximations has a similar character for the TM and TE fields. Therefore in the Neumann problem the second approximation is enough for finding the propagation coefficient of the zero wave in a one-mode waveguide (Figure 4, $d = 10$ and $k = 0.2$).

[31] Now we consider solution of the Neumann problem under small height of the barrier. In this case, expression (18) for the coefficient at the zero wave may be approximately written in the following form:

$$I_0 = 1 + i\frac{\pi}{8}kh\frac{h}{d} = 1 + i0.393kh\frac{h}{d} \quad (19)$$

We compare it to (4a) which is a solution of the similar problem (the difference lies in a curving of the upper wall) by the semi-inversion method. In (4a) the coefficient at the propagating wave has the form

$$I = 1 + i0.395kh\frac{h}{d} \quad (20)$$

One can see that solutions (19) and (20) are almost the same. Therefore the curved upper wall gives almost no input into wave propagation. Thus one can conclude that, solving problems of propagation in the Earth–ionosphere waveguide, the main influence on the field is provided by local irregularities of the Earth surface (especially at the presence of angular points), but not smooth perturbations in the ionosphere.

## 4. Conclusions

[32] Simple expressions are found in this paper for the electromagnetic field in a waveguide with a barrier (such waveguide is the simplest model of the Earth–ionosphere waveguide) under the presence of local singular irregularities at the Earth surface. It is shown that the regularization methods make it possible to form effectively the solution under the presence of singularities in the propagation region. Two methods are considered and a comparison of the obtained solutions is performed.

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