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Nonlinear evolution of a pulse with a linear frequency modulation in a graded-index waveguide

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[1] A weakly nonlinear process of propagation of an electromagnetic pulse with a linear modulation of the carrier frequency in a waveguide structure characterized by different scales of inhomogeneity in the transverse and longitudinal directions is considered. Waveguide channels with such properties are formed, in particular, in the ionospheric F layer due to its stratification under the action of powerful modifying pulses. Thus the method proposed is applicable to a description of the processes of radio wave propagation in the ionosphere with local irregularities. The method also may be used for controlling the parameters (duration, amplitude, etc.) of powerful chirped pulses. Chirps are classified in accordance with the ratio between pulse duration and modulation depth. It is shown that the pulse evolution is determined by three timescales, and analytical relations for chirp parameters are established depending on the transverse and longitudinal inhomogeneity. *INDEX TERMS:* 6984 Radio Science: Waves in plasma; 2487 Ionosphere: Wave propagation; 2439 Ionosphere: Ionospheric irregularities; *KEYWORDS:* Radiowave propagation; pulse evolution; waveguide.

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1. Introduction

[2] Propagation of pulses with a linear deviation of the carrier frequency in dispersive media is accompanied by the effects that make these pulses potentially promising for solution of a number of practical problems. A linear frequency modulation can prove to be a factor that partially counteracts dispersion. For instance, in the linear propagation regime a quadratic phase modulation gives rise to focusing of a pulse in time, that is, up to some distance the pulse compression occurs and only then its dispersion spreading begins [*Akhmanov et al.*, 1988; *Vinogradova et al.*, 1990]. Phase modulation via a more complicated law can lead to splitting of the initial pulse into two separate pulses [*Helczynski et al.*, 2002].

[3] Propagation of powerful probing pulses in the ionosphere leads to excitation of nonlinear effects and formation of waveguide channels [*Molotkov*, 2003; *Molotkov et al.*, 1999]. *Bisyarin and Molotkov* [2002] studied propagation of a short electromagnetic pulse in a graded-index waveguide with a weak longitudinal irregularity. The goal of the work described here was to investigate the process of propagation of a weakly nonlinear pulse using an additional assumption that the carrier frequency depends linearly on time.

[4] The phase of a chirped pulse is expressed as $\Phi = \omega t + \mu\omega^2 t^2$, where parameter μ characterizes the modulation depth. The instantaneous frequency is $\omega + 2\mu\omega^2 t$, and the total variation in the instantaneous frequency of the pulse with duration τ is given by

$$\Delta\omega = 2\mu\omega^2\tau$$

The spectral width $\Delta\Omega$ of the pulse with duration τ is the magnitude of the order of τ^{-1}. Let us compare the modulation depth and spectral width. To this end, we compose the ratio between the total variation in the instantaneous frequency and the spectral width of the pulse as

$$\frac{\Delta\omega}{\Delta\Omega} \sim 2\mu\omega^2\tau^2 \sim \mu\frac{\tau^2}{T^2}$$

where T is the oscillation period, $T = 2\pi/\omega$. In the problem considered here it is assumed that the pulse contains a sufficiently large number of carrier periods, and therefore the ratio between the oscillation period and pulse duration is a small parameter of the problem. By designating this ratio as δ, we get

$$\frac{\Delta\omega}{\Delta\Omega} \sim \frac{\mu}{\delta^2} \tag{1}$$

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Relation (1) allows one to classify chirps according to their depths. The pulses with $\Delta\omega/\Delta\Omega \sim \delta$, i.e., $\mu \sim \delta^3$, will be called chirped pulses. It is these pulses that will be considered in this paper. It is natural to refer to the pulses whose range of instantaneous frequency variation is comparable with the spectral width as strongly chirped pulses. Investigation of their propagation is a separate problem which is beyond the scope of this paper.

2. Simulation of Dynamics of a Chirped Pulse in a Graded-Index Waveguide

[5] Simulation of the process of propagation of a short weakly nonlinear pulse in a graded-index waveguide with a weak longitudinal irregularity was performed similarly to that done by *Bisyarin and Molotkov* [2002]. In dimensionless variables ρ (radial coordinate), s (stretched longitudinal coordinate) and t (time), the model equation acquires the form

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \delta^4 \frac{\partial^2 f}{\partial s^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} -$$

$$(\beta^2(\rho, s, \varphi) + \frac{1}{2}\alpha(\rho, s)\langle f^2\rangle)\frac{\partial^2 f}{\partial t^2} = 0 \qquad (2)$$

The field amplitude f is assumed to be the magnitude of the order of δ; in this case the pulse duration is the magnitude of the order of $1/\delta$. The high-frequency carrier and envelope of the pulse evolve with different phases. For this reason, the envelope phase is given by a separate relation

$$\theta = \frac{Q(s)}{\delta} - \delta t$$

in which the $Q(s)$ function should be defined in the process of the problem solution. It follows from (1) that the term describing the quadratic phase modulation is of the order of δ^3. The solution of (2) is finally sought in the form

$$f = \delta F(\rho, \theta, s, \varphi) \times$$

$$\exp\left[i(\frac{R(s)}{\delta^2} - t - \delta^3 \mu(s) t^2)\right] + \text{complex conjugate} \qquad (3)$$

Note that in a waveguide with a longitudinal irregularity, the modulation coefficient $\mu(s) \sim 1$ depends on the longitudinal coordinate. The complex amplitude F is expanded in a power series of the small parameter

$$F(\rho, \theta, s, \varphi) = F_0(\rho, \theta, s) +$$

$$\delta F_1(\rho, \theta, s) + \sum_{j=2}^{\infty} \delta^j F_j(\rho, \theta, s, \varphi) \qquad (4)$$

If the refractive index is independent of angle φ, the expansion terms of the zero and first orders do not depend on this coordinate as well. Nevertheless, if a spatial bending of the waveguide channel axis is taken into account, the dependence on the azimuthal angle will appear in expansion (4) beginning from the term of the order of δ^2 even in the case of an azimuthally symmetric distribution of the refractive index in the waveguide cross section. The complex amplitude of the wave process is concentrated in the vicinity of the channel axis, and therefore all F_j satisfy the boundary condition $F_j \to 0$ at $\rho \to \infty$.

3. Mode Structure and Pulse Modulation Function

[6] By substituting (3) and (4) into (2) and setting the terms of the same order with respect to δ equal to zero, we get a set of boundary problems for second-order differential equations which, in combination with the conditions of solvability of these problems, allow one to determine successively all the elements of the ansatz.

[7] The $F_0(\rho, \theta, s)$ function is the solution of the Sturm-Liouville problem

$$\frac{\partial^2 F_0}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_0}{\partial \rho} + (\beta^2(\rho, s) - r^2(s))F_0 = 0 \qquad (5)$$

$$\frac{\partial F_0}{\partial \rho}\Big|_{\rho=0} = 0 \qquad F_0\Big|_{\rho\to\infty} \to 0$$

Bisyarin and Molotkov [2002] have demonstrated the solvability of this problem for a sufficiently wide and practically important class of the $\beta(\rho, s)$ functions. Let $r^2(s)$ be the eigenvalue of the problem (5) and let $V(\rho, s)$ be the eigenfunction corresponding to this eigenvalue and normalized by the condition

$$\int_0^{\infty} \rho V^2(\rho, s) d\rho = 1 \qquad (6)$$

[8] The complex amplitude in the principal order can be presented as a product

$$F_0(\rho, \theta, s) = V(\rho, s) U(\theta, s)$$

where $U(\theta, s)$ describes the envelope of the selected mode in the main approximation. Below we shall refer to it as the pulse envelope.

[9] The first-order correction in expansion (4) satisfies the equation

$$\frac{\partial^2 F_1}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_1}{\partial \rho} + (\beta^2(\rho, s) - r^2(s))F_1 =$$

$$2i(\beta^2(\rho, s) - r(s)Q'(s))\frac{\partial F_0}{\partial \theta} -$$

$$2Q(s)(2\beta^2(\rho, s)\mu(s) + r(s)\mu'(s)Q(s))F_0$$

and the same boundary conditions as function F_0. The solution of this problem has the structure that makes it possible to express the dependences on phase θ via functions U and $\partial U/\partial\theta$

$$F_1(\rho,\theta,s) = 2iW_1(\rho,s)\frac{\partial U}{\partial\theta} - 2Q(s)W_2(\rho,s)U(\theta,s)$$

[10] Here the pair of functions W_1 and W_2 is defined as solutions of inhomogeneous equations

$$\frac{\partial^2 W_1}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial W_1}{\partial\rho} + (\beta^2(\rho,s) - r^2(s))W_1 =$$

$$(\beta^2(\rho,s) - r(s)Q'(s))V$$

$$\frac{\partial^2 W_2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial W_2}{\partial\rho} + \left(\beta^2(\rho,s) - r^2(s)\right)W_2 =$$

$$\left(2\beta^2(\rho,s)\mu(s) + r(s)\mu'(s)Q(s)\right)V$$

that satisfy the boundary conditions in problem (5). If for the right-hand sides of these equations the relations

$$\int_0^\infty \rho\left(\beta^2(\rho,s) - r(s)Q'(s)\right)V^2(\rho,s)d\rho = 0$$

$$\int_0^\infty \rho\left(2\beta^2(\rho,s)\mu(s) + r(s)\mu'(s)Q(s)\right)V^2(\rho,s)d\rho = 0$$

are fulfilled, such functions W_1 and W_2 exist. This allows one, with due account of the condition of normalization (6), to express the envelope phase and modulation function through the eigenvalue and the eigenfunction of the problem (5)

$$Q'(s) = r(s) + \frac{1}{r(s)}\int_0^\infty \rho\left(\frac{\partial V}{\partial\rho}\right)^2 d\rho \qquad (7)$$

$$\mu(s) = \frac{\mu_0}{Q^2(s)} \qquad \mu_0 = \text{const} \qquad (8)$$

Equations (7) and (8) establish the relation between the modulation function and the $Q(s)$ and $R(s)$ functions that determine phases of the envelope and high-frequency carrier of the propagating mode. The modulation function is therefore related to these functions and cannot be specified in an arbitrary manner. It varies in accordance with (8) as the pulse propagates in a longitudinally inhomogeneous waveguide.

4. Envelope of a Chirped Pulse

[11] The second-order correction to the complex amplitude F_2 satisfies the inhomogeneous equation

$$\frac{\partial^2 F_2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F_2}{\partial\rho} + \left(\beta^2(\rho,s) - r^2(s)\right)F_2 =$$

$$2i\left(\beta^2(\rho,s) - r(s)Q'(s)\right)\frac{\partial F_1}{\partial\theta} -$$

$$2Q(s)\left(2\beta^2(\rho,s)\mu(s) + r(s)\mu'(s)Q(s)\right)F_1 +$$

$$\left(\beta^2(\rho,s) - Q'^2(s)\right)\frac{\partial^2 F_0}{\partial\theta^2} +$$

$$4i\beta^2(\rho,s)\mu(s)Q(s)\frac{\partial F_0}{\partial\theta} +$$

$$4\beta^2(\rho,s)\mu(s)\left(\theta - \mu(s)Q^2(s)\right)F_0 -$$

$$2ir(s)\frac{\partial F_0}{\partial s} + 2i\mu'(s)Q^2(s)Q'(s)\frac{\partial F_0}{\partial\theta} -$$

$$ir'(s)F_0 + \mu'(s)Q(s)\left(4\theta r(s) + \mu'(s)Q^3(s)\right)F_0 -$$

$$\alpha(\rho,s)|F_0|^2 F_0$$

and the boundary conditions $\partial F_2/\partial\rho|_{\rho=0} = 0$ and $F_2 \to 0$ at $\rho\to\infty$. The condition of solvability of this problem implies the equation for the pulse envelope $U(\theta,s)$

$$2ir(s)\frac{\partial U}{\partial s} + g(s)\frac{\partial^2 U}{\partial\theta^2} + ij(s)\frac{\partial U}{\partial\theta} + ir'(s)U +$$

$$4\theta\mu(s)r(s)Q'(s)U + d(s)U + h(s)|U|^2U = 0 \qquad (9)$$

The coefficients of the equation are given by

$$g(s) = 4\int_0^\infty \rho\left(\beta^2(\rho,s) - r(s)Q'(s)\right)V(\rho,s)W_1(\rho,s)d\rho -$$

$$\int_0^\infty \rho\left(\beta^2(\rho,s) - Q'^2(s)\right)V^2(\rho,s)d\rho$$

$$h(s) = \int_0^\infty \rho\alpha(\rho,s)V^4(\rho,s)d\rho$$

$$j(s) = 4Q(s)\int_0^\infty \rho\left(\beta^2(\rho,s)-\right.$$

$$\left.r(s)Q'(s)\right)V(\rho,s)W_2(\rho,s)d\rho+$$

$$8Q(s)\mu(s)\int_0^\infty \rho\left(\beta^2(\rho,s)-\right.$$

$$\left.r(s)Q'(s)\right)V(\rho,s)W_1(\rho,s)d\rho-$$

$$4Q(s)\mu(s)\int_0^\infty \rho\left(\beta^2(\rho,s)-Q'^2(s)\right)V^2(\rho,s)d\rho$$

$$d(s) = 4\mu_0\mu(s)\int_0^\infty \rho\left(\beta^2(\rho,s)-Q'^2(s)\right)V^2(\rho,s)d\rho$$

The dependence of the coefficients of (9) on variable s manifests the influence of the longitudinal irregularity of the waveguide channel on evolution of the pulse envelope and, in particular, on its linear frequency modulation.

[12] Solution of (9) in a general case can be performed by the method described in the books by *Molotkov* [2003] and *Molotkov et al.* [1999]. As a first step, it is necessary to solve a simplified model problem, which will lead to the conclusions on a qualitative behavior of the solution. Then this solution can be used as a structural basis of the ansatz for the solution of the complete problem.

[13] To simplify the problem (9), we impose additional restrictions on the longitudinal inhomogeneity of the waveguide. We write the sought function U in the form

$$U(\theta,s) = \frac{u(\theta,s)}{\sqrt{r(s)}}e^{i\theta}$$

extracting explicitly an exponential multiplier with the envelope phase and, similarly to *Bisyarin and Molotkov* [2002], the multiplier that characterizes amplitude variations as functions of the longitudinal coordinate. Let us suppose that the $u(\theta,s)$ function defined in such a way depends on the variable

$$x = \theta - \int_0^s \frac{2g(s')+j(s')}{2r(s')}ds'$$

alone, which can be achieved if the coefficients of (9) are related by

$$d(s)-j(s)-g(s)+$$

$$4\mu(s)r(s)Q'(s)\int_0^s \frac{2g(s')+j(s')}{2r(s')}ds' = 0$$

$$-4\mu(s)r(s)Q'(s) = g(s)$$

$$g(s)r(s) = h(s)$$

It is these relations that present additional assumptions on the longitudinal irregularity of the waveguide. Under these conditions the $u(x)$ function obeys the second Painlevé equation

$$u'' - xu + u^3 = 0 \qquad (10)$$

This equation is well studied from the point of view of existence of moving critical points in the solutions [see *Golubev*, 1950; *Kudryashov*, 2004, and references therein]; a relation with linear integral equations of definite kinds has been investigated in detail by *Ablowitz et al.* [1978, 1980]. *Gibbon et al.* [1985] has proved that it is possible to construct N-soliton solutions of nonlinear differential equations in partial derivatives if they possess the Painlevé property. Reduction of the second Painlevé equation to a linear integral equation and the proof of the existence of bounded solutions were given by *Ablowitz and Segur* [1977].

[14] In our work, the numerical solution of (10) was performed by the Runge-Kutta method. Figure 1 shows the graph of this solution (curve 1). For the sake of a comparison, Figure 1 presents a sech soliton with the same amplitude (curve 2) and the Airy function (curve 3). The soliton of the nonlinear Schrödinger equation is the solution of a similar problem [*Bisyarin and Molotkov*, 2002] without frequency modulation. The Airy function is presented for comparison because linearization of (10) is the Airy equation. The plotted solution has a localized character, that is, it tends to zero at $|x| \to 0$. Comparison with the sech soliton leads to the conclusion that a linear frequency modulation of the high- frequency carrier of the pulse gives rise to formation of an oscillating tail of the envelope and leads to a greater steepness of the leading edge of the pulse.

5. Conclusions

[15] Propagation of a weakly nonlinear short pulse with a linear frequency modulation in a graded-index waveguide with a weak longitudinal irregularity has been investigated on the basis of a nonlinear wave equation. Depending on the ratio between the spectral width of the pulse and the frequency modulation depth, the pulses were classified as chirped and strongly chirped. The work was devoted to the detailed investigation of propagation of chirped pulses, that is, the pulses whose modulation depth is much less than the spectral width.

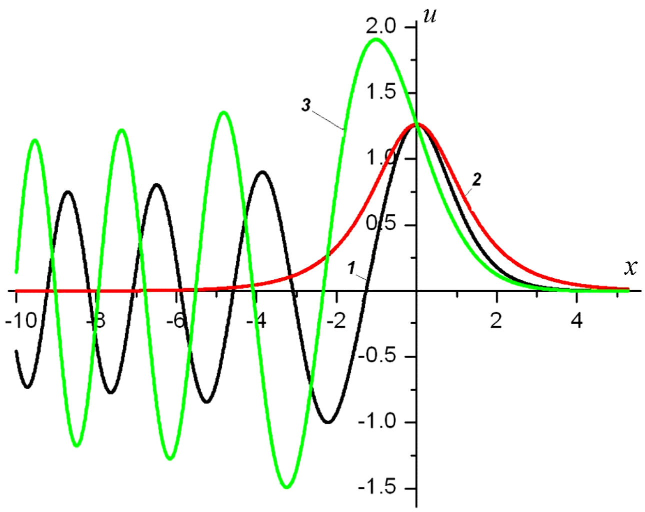

Figure 1. Solution of (10) (curve 1) in comparison with the soliton of the nonlinear Schrödinger equation (curve 2) and Airy function $Ai(x)$ (curve 3).

[16] The model nonlinear wave equation was solved by using an asymptotic procedure, whose small parameter was the order of magnitude of the pulse amplitude. The longitudinal irregularity of the waveguide had the scale of the order of a square of the small parameter, and the magnitude of the phase modulation of the chirped pulse was proportional to the third power of this parameter. The use of ansatz (3) and (4) for the solution of the nonlinear wave equation allows one to detach in a natural manner the linear Sturm-Liouville problem (5), whose eigenvalue defines the propagation constant of the high-frequency carrier as a function of the longitudinal coordinate, and its eigenfunction describes evolution of the transverse distribution of the pulse field as it propagates in the graded-index waveguide.

[17] Propagation of a pulse is a nonlinear process characterized by three velocities. A fast process, i.e., propagation of a high-frequency carrier, is modulated by the envelope whose evolution has two scales and is formed by the evolution of the envelope phase with a medium velocity and a slow amplitude variation. The pulse envelope is described by (9) which in a special case can be reduced to the equation of the Painlevé class (10). Numerical solution of (10) has shown that a chirped pulse has a steeper leading edge as compared with a soliton pulse in the absence of frequency modulation, and an oscillating, but decaying, tail of the envelope is formed at the trailing edge.

[18] In the process of a successive realization of the asymptotic procedure, an important relationship (7) and (8) between the phase of the high-frequency carrier, envelope phase, and modulation coefficient of the pulse has been established. The physical consequence of this relationship is that the chirp cannot be specified in an arbitrary manner in a graded-index waveguide channel; it must be tailored to the parameters of the transverse and longitudinal irregularities of the waveguide. The result obtained in our work explains this fact and provides a suitable tool for exploitation of chirped pulses.

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