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Normal waves of the anisotropic Earth–ionosphere waveguide in the VLF-ULF range

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[1] The problem on determination of the electromagnetic field of a near-Earth horizontal electric dipole in the frequency range $10^{-1} - 10^3$ Hz at distances $10^{-1} - 10^3$ km in the Earth–ionosphere waveguide has been solved. The ionosphere is assumed to be horizontally stratified and anisotropic, and the Earth is assumed to be homogeneous to some depth and infinitely conducting below. The representation of the electromagnetic field as expansion in waveguide modes is given. The zero mode is regarded as anisotropic. Local modes are given in the isotropic approximation of the anisotropic ionosphere due to inclusion of an effective exponential conductivity profile. An analytical description of the field reflection for vertical and horizontal polarizations has been obtained for this model. At short distances from the source, the solution coincides with the known solution for the case of the absence of the ionosphere. The range of distances beginning from which the influence of the ionosphere on the near-Earth magnetic field manifests itself has been found. *INDEX TERMS:* 6964 Radio Science: Radio wave propagation; 2487 Ionosphere: Wave propagation; 2439 Ionosphere: Ionospheric irregularities; *KEYWORDS:* VLF-ULF propagation; ionospheric waveguide; dipole-electromagnetic field.

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1. Introduction

[2] The paper is concerned with the problem of calculation of the electromagnetic field of a near-Earth horizontal electric dipole in the frequency range below 1 kHz at distances less than 1 Mm in the Earth–ionosphere waveguide. The waveguide is formed by the Earth characterized by the conductivity and the ionosphere. The electrical properties of the ionosphere are determined by the vertical profiles of the electron density and effective collision frequency of electrons with neutral particles and by the Earth's magnetic field. At heights 30–80 km, the ionospheric conductivity has to be specified. In the frequency range 0.1–5 Hz, it is required that the ion collision frequency with neutral particles and the profiles of the positive ion mean mass be additionally specified. The upper wall of the waveguide formed by the ionosphere is, therefore, anisotropic and horizontally inhomogeneous. The waveguide inhomogeneity is mainly due to the variability in the electrical properties of the ionosphere on transition from night to day. Some contribution to the waveguide inhomogeneity comes from the dependence of the Earth's magnetic field on the horizontal coordinate. Owing to the restriction of the field calculation domain to 1 Mm, the homogeneous waveguide model can be used since there is little probability that the day-night transition will fall into this domain and the dependence of the electrical properties of the ionosphere on the Earth's magnetic field may be neglected because of its weak variability within a distance of 1 Mm along the Earth's surface. However, in the general case the effect of sphericity on the field in the vicinity of the source cannot be neglected. Indeed, the vertical component of the electric field E_r of a vertical electric dipole P_0 in a homogeneous spherical cavity of height h in the one-mode approximation has the form [*Watson*, 1919]

$$E_r = -\frac{\nu(\nu+1)P_0}{4\varepsilon_0 a^2 h}\frac{P_\nu[\cos(\pi-\theta)]}{\sin(\nu\pi)}$$

where ε_0 is the dielectric permeability of the vacuum, a is the Earth's radius, and $P_\nu(x)$ is the Legendre function [*Bateman and Erdelyi,* 1953]. Symbol ν is related to the propagation constant S by $\nu(\nu+1) = k^2a^2S^2$, $k = \omega/c$ being the wave number in vacuum. The propagation constant S having the dependence on time of the form $\exp(-i\omega t)$ is related to the ionospheric conductivity σ_i as $S^2 = 1 + [(1+i)/h\sqrt{2\omega\mu_0\sigma_i}]$, where μ_0 is the magnetic permeability of the vacuum.

[3] In a plane waveguide, the vertical component of the vertical electric dipole P_0 is expressed through the cylindrical

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function $H_0^{(1)}(kSa\theta)$ [*Jahnke et al.*, 1960]

$$E_r^p = \frac{ik^2 S^2 P_0}{4\varepsilon_0 h} H_0^{(1)}(kSa\theta) \quad (1)$$

[4] To describe the field in the neighborhood of the source, the Legendre function can be substituted by its expression which can be used if its argument is in the vicinity of " − 1" [*Bateman and Erdelyi*, 1953]

$$\frac{P_\nu[\cos(\pi-\theta)]}{\sin\nu\pi} = \frac{1}{\pi}[2C_e + 2\psi(\nu+1) +$$

$$\pi\cot\nu\pi + 2\ln(\sin\theta/2)]$$

where $\psi(\nu+1)$ is the logarithmic derivative of the Gamma function, and $C_e = -\psi(1)$ is the Euler constant. Similarly, the Legendre function can be replaced by its expression with small arguments [*Jahnke et al.,* 1960], which allows comparison of the vertical component in a spherical waveguide with its plane approximation (1)

$$E_r - E_r^p = -\frac{\nu(\nu+1)P_0}{4\pi\varepsilon_0 a^2 h}[i\pi + 2\psi(\nu+1) +$$

$$\pi\cot\nu\pi - \ln\nu(\nu+1)]$$

Near the source their difference is a frequency-dependent constant. In the vicinity of the first Schumann frequencies this constant becomes high. As frequency increases, the constant tends to zero because of a complexity of ν, which occurs in the Earth–ionosphere waveguide at frequencies above 100 Hz. The tangential components of the magnetic field and the electromagnetic field of a horizontal electric dipole in the vicinity of the source can be described by the plane waveguide approximation.

[5] At distances ρ exceeding a triple effective waveguide height h, only the field of one leading quasi-TEM mode is left in the electromagnetic field of a point source [*Madden and Thompson*, 1965; *Wait*, 1962]. This mode is found by numerical integration of a system of ordinary differential equations following from the Maxwell equation [*Galyuk and Ivanov*, 1978; *Hynninen and Galyuk*, 1972; *Jones*, 1967; *Kirillov and Kopeykin*, 2002, 2003; *Kirillov et al.*, 1997; *Krasnushkin and Yablochkin*, 1963; *Madden and Thompson*, 1965; *Pappert and Miler*, 1974] or analytically [*Greifinger and Greifinger*, 1978, 1979, 1986; *Kirillov*, 1972, 1978, 1993, 1996, 1998; *Kirillov and Pronin,* 1974]. Only numerical calculations yield the results corresponding to the real model of the ionosphere. However, the analytical consideration of the reflection from the ionosphere allows understanding of the results obtained in numerical calculations and is suitable by itself for a simple exponential model of the ionosphere describing adequately the actual situation in some cases. In the analytical description which will be given below for the isotropic model of conductivity, two complex heights are introduced, i.e., the capacitance height

$$h_C = \int_0^\infty \frac{dz}{(1+z/a)^2\,[1+i\sigma_i/(\omega\varepsilon_0)]}$$

which plays the role of a normalization integral for the leading TEM mode and inductance height h_L. By the real part, the capacitance height is always lower than the inductance one. In the exponential model of the ionospheric conductivity, $\mathrm{Re}h_C$ is approximately equal to the height at which the conduction currents are equal to the displacement currents, and an approximate estimate of the height h_C itself is obtained from the equation $i\sigma(h_C)/(\omega\varepsilon_0) = 1$ [*Greifinger and Greifinger,* 1978; *Kirillov*, 1972, 1978; *Kirillov and Pronin*, 1974]. The upper inductance height h_L is deduced in this case from [*Kirillov*, 1972, 1978; *Kirillov and Pronin*, 1974]

$$\frac{\sigma(h_L)}{\omega\varepsilon_0} = i\left(\frac{\alpha}{k\gamma}\right)^2 \quad (2)$$

where $\alpha = d/dz \ln\sigma_i$. *Greifinger and Greifinger* [1978] also considered two similar heights, i.e., h_1 and h_2, but the equation for the second height included in its right-hand side $\alpha/2k$ instead of α/ky, which led to a systematic error of the order of 1 km in its determination.

[6] With two heights introduced above, the surface impedance of the ionosphere at the level of the Earth's surface at $|kh_C| \ll 1$ can be written as

$$\delta_i = -ik(h_L - h_C S^2) \quad (3)$$

where S is the sine of the wave incidence angle near the Earth's surface. With this ionospheric impedance and ideally reflecting Earth, the eigenvalue (propagation constant S) is given by $S^2 = h_L/h_C$.

[7] In the general case of an anisotropic ionosphere, the inductance height h_L is a matrix expressed through the ionospheric impedance at the Earth's surface level at normal wave incidence $h_L = i\delta_i(0)/k$. This matrix is obtained by numerical integration over the region of the ionosphere responsible for radio wave reflection. The region extends to 2 Mm in the frequency range 0.1–5 Hz [*Kirillov and Kopeykin*, 2002, 2003]. For the plane homogeneous model of the waveguide, the two heights introduced above allow one to express the electromagnetic field of a horizontal electric dipole at the Earth's surface through the first function of the zero order and its derivatives [*Kirillov and Kopeykin,* 2002]. In the general case these expressions are rather cumbersome. Below we give the fields for the anisotropic ionosphere but exhibiting an azimuthal symmetry, which is realized for the real Earth–ionosphere waveguide at frequencies above 5 Hz:

$$E_z = -\frac{iIlkSZ_0\delta_g}{4h_C}\left(\cos\varphi + \frac{h_{L,xy}}{h_{L,xx}}\sin\varphi\right)\dot{H}_0^{(1)}(\xi)$$

$$H_z = -\frac{iIlkS\delta_g^2}{4h_C}\frac{h_{L,xy}}{h_{L,xx}}\left(\cos\varphi + \frac{h_{L,xy}}{h_{L,xx}}\sin\varphi\right)\dot{H}_0^{(1)}(\xi) \quad (4)$$

$$H_\varphi = -\frac{Ilk\delta_g}{4h_C}\Big\{\Big[H_0^{(1)}(\xi) +$$

$$\left(1+\frac{h_{L,xy}^2}{h_{L,xx}^2}\right)\xi^{-1}\dot{H}_0^{(1)}(\xi)\cos\varphi +$$

$$\left.\left.\frac{h_{L,xy}}{h_{L,xx}} H_0^{(1)}(\xi)\sin\varphi\right]\right\}$$

$$H_\rho = \frac{Ilk\delta_g}{4h_C}\left\{\left[\xi^{-1}\dot{H}_0^{(1)}(\xi)+\right.\right.$$

$$\left.\left.\frac{h_{L,xy}^2}{h_{L,xx}^2}\left[H_0^{(1)}(\xi)+\xi^{-1}\dot{H}_0^{(1)}(\xi)\right]\right]\sin\varphi+\right.$$

$$\left.\frac{h_{L,xy}}{h_{L,xx}} H_0^{(1)}(\xi)\cos\varphi\right\}$$

$$E_\rho = -Z_0\delta_g H_\varphi \qquad E_\varphi = Z_0\delta_g H_\rho$$

$$S^2 = \frac{1}{h_C(h_L^{-1})_{xx}}$$

where Il is the current moment of the horizontal dipole oriented along the x axis of the Cartesian coordinate system $\{x, y, z\}$. Angle φ is counted off from this axis in the direction of y, and ρ is the distance from the dipole. The elements of the matrix inductance height h_L satisfy the conditions $h_{L,xy} + h_{L,yx} = 0$ and $h_{L,xx} = h_{L,yy}$. The point above the function implies differentiation with respect to its argument. For $|k\rho| \ll 1$, expression (4) transforms into

$$E_z = \frac{IlZ_0\delta_g}{2\pi\rho}\left(\cos\varphi + \frac{h_{L,xy}}{h_{L,xx}}\sin\varphi\right)$$

$$H_z = \frac{h_{L,xy}}{h_{L,xx}}\delta_g Z_0^{-1}E_z$$

$$-H_\varphi = \frac{iIl\delta_g}{2\pi k\rho^2 h_{L,xx}}\cos\varphi$$

$$H_\rho = \frac{iIl\delta_g}{2\pi k\rho^2 h_{L,xx}}\sin\varphi$$

The magnetic field in the vicinity of the horizontal electric dipole is independent of the capacitance height h_C.

[8] It should be noted that in the general case of the ionosphere anisotropy, when the dielectric permeability tensor in the region responsible for radio wave reflection depends on the vertical component of the Earth's magnetic field and also exhibits an appreciable dependence on its horizontal components, the propagation constant enters the function argument with a multiplier $\cos\alpha$. Angle α is the angle between the beam direction coinciding in a homogeneous medium with the direction from a transmitter to a receiver and the direction of the horizontal wave propagation gradient [*Born and Wolf,* 1964]. The general anisotropy is realized in the Earth–ionosphere waveguide at frequencies below 5 Hz. *Greifinger and Greifinger* [1986] did not take into account the multiplier $\cos\alpha$ in the argument of the function in the case of the general ionosphere anisotropy, which is incorrect.

[9] At short distances from the source, not exceeding a half of the effective waveguide height, the ionosphere does not affect the near-Earth electromagnetic field. In geoelectrical prospecting, in the case of a homogeneous model of the ground conductivity, the horizontal components of the magnetic field at $k\rho \ll 1$ are calculated using the formulae of *Baños* [1966], *Fok and Bursian* [1926], *Bursian* [1972], *Vanyan* [1997], and *Wait* [1961]

$$-H_\varphi = \frac{iIl\cos\varphi}{4\rho^2}\dot{J}_0(\xi)\dot{H}_0^{(1)}(\xi) \tag{5}$$

$$H_\rho = \frac{iIl\sin\varphi}{4\rho^2}\left[3\dot{J}_0(\xi)\dot{H}_0^{(1)}(\xi)+\right.$$

$$\left.\xi\left(\dot{J}_0(\xi)H_0^{(1)}(\xi) + J_0(\xi)\dot{H}_0^{(1)}(\xi)\right)\right]$$

Here, $\xi = k\rho\sqrt{\varepsilon_g'}/2$, where ε_g' is the complex relative dielectric permeability of the ground $\varepsilon_g' = \varepsilon_g + (i\sigma_g/\omega\varepsilon_0)$. At such short distances the observation point is close to the source being no longer an elementary dipole. The electromagnetic field of a distributed source is obtained by summing the fields of the dipoles into which the source is divided according to the current distribution through it.

[10] In the case

$$|\xi| \ll 1 \text{ and } |\varepsilon_g'| \gg 1$$

$$-H_\varphi = \frac{Il\cos\varphi}{4\pi\rho^2} \qquad H_\rho = \frac{Il\sin\varphi}{4\pi\rho^2} \tag{6}$$

The magnetic field is independent of the ground conductivity and decreases inversely proportionally to the square of the distance from the source. At $|\xi| \gg 1$, i.e., at the distance from the source equal to several Earth's skin layers, the expression for the field acquires the form

$$-H_\varphi = \frac{iIl\delta_g\cos\varphi}{2\pi k\rho^3}$$

$$H_\rho = \frac{iIl\delta_g\sin\varphi}{\pi k\rho^3} \tag{7}$$

The magnetic field decreases in this zone inversely proportionally to the cube of the distance from the source. By the modulus, the longitudinal component H_ρ turns out to be twice as high as the component H_φ. At intermediate distances $h/2 < \rho < 2h$ there is a transition region, at the lower boundary of which the electromagnetic field is nearly independent of the ionosphere and whose upper boundary coincides with the field of the only waveguide wave that propagates (zero normal wave). This paper describes the electromagnetic field in the transition region. This problem has been considered and solved by *Bannister and Williams* [1974], *Galeys* [1968], and *Wait* [1962] by the method of summation of jumps in the case of an ideally reflecting ionosphere of height h. According to *Bannister and Williams* [1974], the horizontal components of the magnetic field in the transition region are given by

$$-H_\varphi = \frac{iIl\delta_g \cos\varphi}{2\pi k\rho^3} G(t)$$

$$H_\rho = \frac{iIl\delta_g \sin\varphi}{2\pi k\rho^3} H(t) \tag{8}$$

where

$$G(t) = \frac{2t}{\pi}\coth(t) + (1-\frac{2}{\pi})t^2\sinh^{-2}(t)$$

$$H(t) = G(t) + t^3\sinh^{-2}(t)\coth(t)$$

and

$$t = \frac{\pi\rho}{2\mathrm{Re}\ h_L}$$

[11] In contrast to *Bannister and Williams* [1974], the height $\mathrm{Re}h_L$ is used here instead of h because this height corresponds most of all to the wave reflection height. The solution of the problem on the field in the transition region given by *Bannister and Williams* [1974] has a significant drawback. At low t, the formulae for the field transform into the expressions describing the field decrease inversely proportional to the cube of the distance from the source ($\approx \rho^{-3}$) derived for the case of the absence of the ionosphere. At high t the formulae given above transform into the expressions for the quasi-static field of the leading normal wave in an isotropic waveguide. Actually, depending on the parameters of the problem, i.e., the frequency and ground conductivity, even at $t \ll 1$ the distance to the source can prove to be less or of the order of the Earth's skin layer, and then the field decreases as ρ^{-2}. In the upper part of the transition region at frequencies below 10 Hz, the anisotropy of the medium has to be taken into account. Moreover, the upper boundary of the region is not always in the static zone. The use of the model with an effective height as the ionosphere model is not justified as well.

[12] It is worth noting that *Bannister* [1986] expanded the transformation of the expression for the field into the expression for the zero normal wave to the case when the upper boundary of the transition region falls into the wave zone. The problem on the electromagnetic field in the transition region was also considered by *Saraev and Kostin* [1998]. They presented the field in the form of weakly converging integrals of Bessel functions, and the ionosphere model was described by a homogeneous isotropic conductivity.

[13] Thus the problem of description of the field in the transition region is still important. Below, this problem is solved by representing the field as a sum of normal waves, including local ones.

Table 1. Profile Parameters

Conditions	z, km	$\sigma(z)$, 10^{-9} S m^{-1}	$\frac{d}{dz}\ln\sigma$, km^{-1}
Day	50	2.08	0.29
Night	75	2.03	0.35

2. Waveguide Model

[14] The ground is characterized by a constant conductivity σ_g. The ionosphere has in its lower part an exponential conductivity profile $\sigma(z)$ which is different for day and night. The profile parameters are listed in Table 1.

[15] *Mushtak and Williams* [2002] also characterize the conductivity model by an exponent but having different parameters above and below a height of 55 km, which is likely to describe more adequately the lower ionosphere. The representation of the ionosphere as an exponential conductivity profile leads to a logarithmic dependence of the real part of the capacitance height on frequency [*Greifinger and Greifinger,* 1978; *Kirillov and Pronin*, 1974; *Kirillov et al.*, 1997]

$$\mathrm{Re}\ h_C(f) = \mathrm{Re}\ h_C(f_1) + \alpha_C^{-1}\ln(f/f_1)$$

where α_C^{-1} is the scale of variations in the ionospheric conductivity profile. The imaginary part of the capacitance height is negative; it is related to the scale α_C^{-1} and is independent of frequency $\mathrm{Im}h_C = -\pi/(2\alpha_C)$. From the conductivity profile with the parameters given in Table 1, we obtain for a frequency of 50 Hz the following capacitance heights: $h_C = (51.0 - i5.4)$ km and $h_C = (75.9 - i4.5)$ km for day and night, respectively.

[16] The frequency model of the matrix local ionospheric inductance for four models of the ionosphere (day and night at low and high solar activity levels) was presented by *Kirillov and Kopeykin* [2003]. The ionosphere model specifies profiles of electron density, effective collision frequency of electrons and ions with neutral particles, and the average mass of positive ions. In the frequency range 0.1–5 Hz, the region responsible for reflection extends to a height of the order of 2 Mm. At lower frequencies the function $\tilde{h}_L = \sqrt{\mathrm{Det}h_L}$ is a convenient isotropic approximation of the matrix local inductance. The graphs of this function versus frequency for two different conditions are given in Figures 1–4.

[17] In addition, $\tilde{h}_L$ is used to synthesize the effective exponential conductivity profile of the ionosphere, so $\mathrm{Re}\tilde{h}_L$ is considered to be equal to the real part of the inductance height at which the conductivity is $\sigma = \omega\varepsilon_0(\alpha/(k\gamma))^2$ and the scale of variations in the conductivity profile is $\alpha = \pi/(2\mathrm{Im}\tilde{h}_L)$. The effective conductivity profile is used only to describe local modes which are treated in the isotropic approximation alone. The use of the effective conductivity profile for the description of the local inductance as a function frequency is similar to that presented by *Mushtak and Williams* [2002]. Parameters of the effective profile of conductivity are obtained by some approximation of the anisotropic ionosphere.

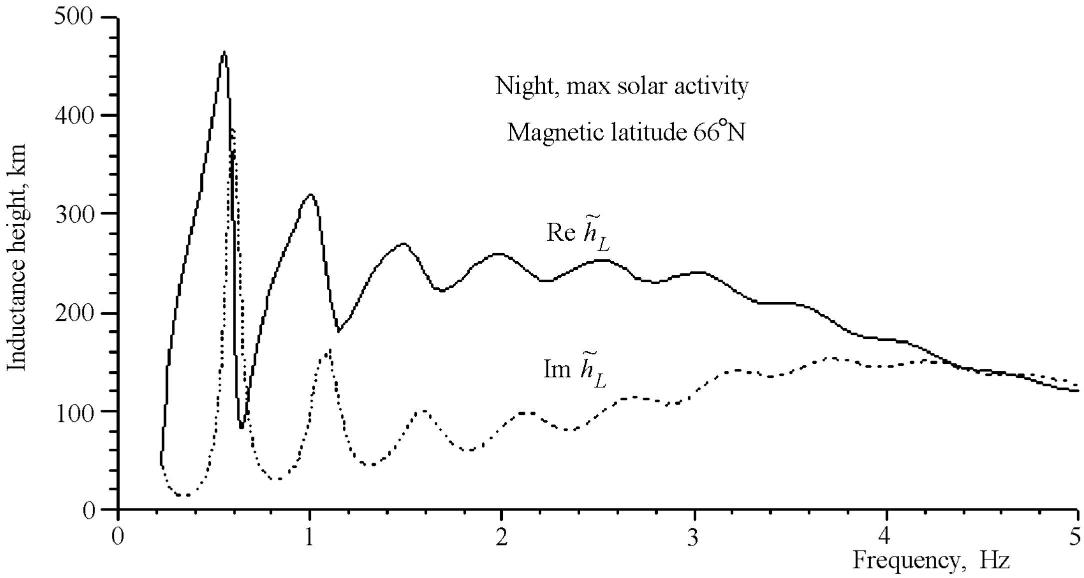

Figure 1. Dependence of $\mathrm{Re}\,\tilde{h}_L$ and $\mathrm{Im}\,\tilde{h}_L$ on frequency under nighttime conditions. Frequencies are from 0.2 to 5 Hz.

3. Horizontally Polarized Modes

[18] We assume that $\partial/\partial x = ikS$ and $\partial/\partial y = 0$. Then for the horizontally polarized electromagnetic field, the system of Maxwell equations yields the following system of ordinary differential equations with respect to the vertical z coordinate:

$$\frac{1}{ik}\frac{d}{dz}E_y = -Z_0 H_x$$

$$\frac{1}{ik}\frac{d}{dz}(-Z_0 H_x) = (\varepsilon' - S^2)E_y \tag{9}$$

$$Z_0 H_z = S E_y$$

$$E_z = Z_0 H_y = E_x = 0$$

which is reduced to one second-order differential equation for the E_y component

$$\left[\frac{d^2}{dz^2} + k^2(\varepsilon' - S^2)\right] E_y = 0$$

In the case of the exponential conductivity profile, when $d/dz \ln(\varepsilon' - 1) = \alpha$ is independent of z, this equation is reduced exactly to the Bessel equation

$$\left(\xi^{-1}\frac{d}{d\xi}\xi\frac{d}{d\xi} + 1 + \frac{\mu^2}{\xi^2}\right) E_y = 0$$

$$\xi = \frac{2k}{\alpha}\sqrt{\varepsilon' - 1}$$

$$\mu = 2kC/\alpha$$

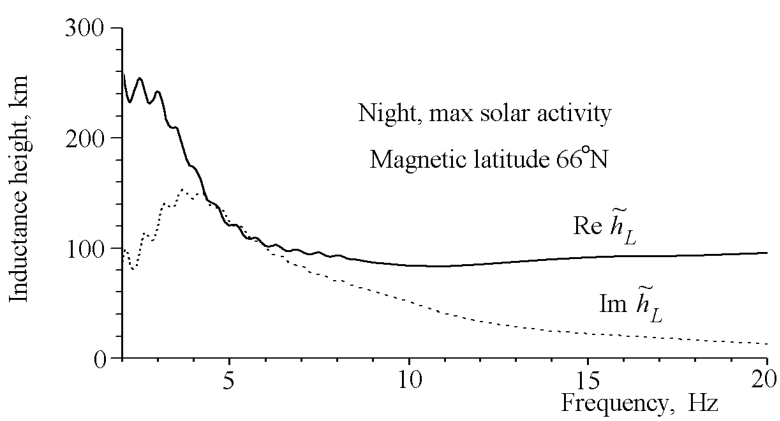

Figure 2. Dependence of $\mathrm{Re}\,\tilde{h}_L$ and $\mathrm{Im}\,\tilde{h}_L$ on frequency under nighttime conditions. Frequencies are from 2 to 20 Hz.

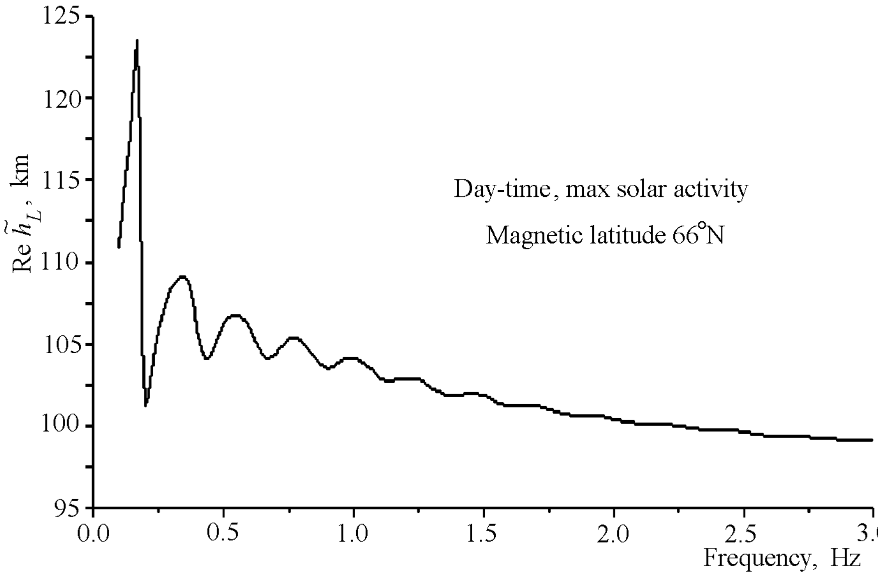

Figure 3. Dependence of Re $\tilde{h}_L$ on frequency under daytime conditions.

$$S^2 + C^2 = 1$$

Its solution satisfying the principle of a sufficient field decrease at $z \to \infty$ is the first function $H^{(1)}_{-i\mu}(\xi)$.

[19] At the Earth's surface, $|\xi| \ll 1$. The ionospheric reflection coefficient at the Earth's surface is given by

$$E_y(0) = A(1 + R_i^{\perp})$$

$$-Z_0 H_x(0) = CA(1 - R_i^{\perp})$$

By representing $H^{(1)}_{-i\mu}(\xi)$ through Bessel functions $J_{i\mu}(\xi)$ and $J_{-i\mu}(\xi)$ [*Jahnke et al.*, 1960], we get for the ionospheric reflection coefficient $R_i^{\perp}$ at the Earth's surface

$$R_i^{\perp}(C) = R_{h_L}^{\perp} \exp(2ikh_L C)$$

where

$$-R_{h_L}^{\perp}(C) = \frac{\Gamma(1 + i\mu)}{\Gamma(1 - i\mu)} \exp(2iC_e\mu)$$

$C_e = \ln\gamma$ is the Euler constant, and $R_{h_L}^{\perp}$ is the reflection coefficient referred to the complex height h_L derived from the equation $1 - \varepsilon'(h_L) = [\alpha/(k\gamma)]^2$. Here and in below, the

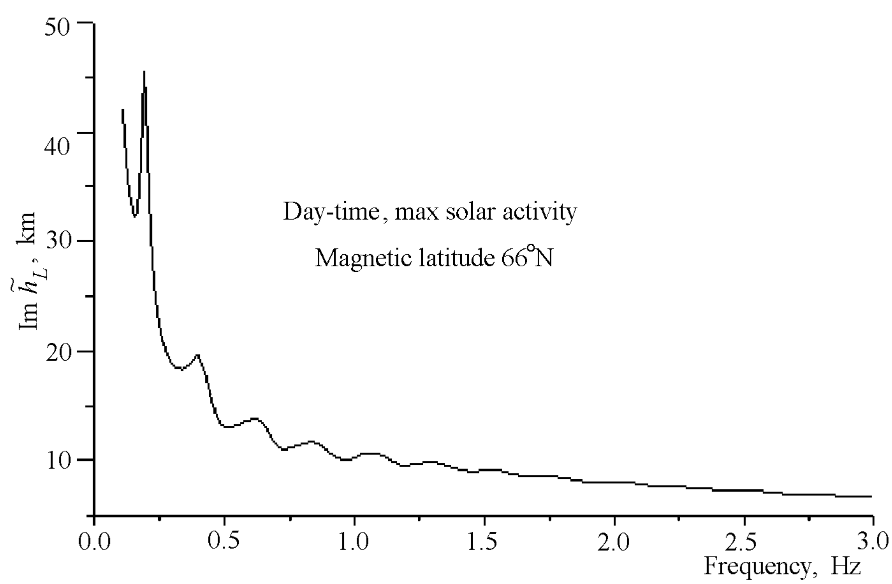

Figure 4. Dependence of Im $\tilde{h}_L$ on frequency under daytime conditions.

sign "∼" above height parameters of the isotropic conductivity profile $\tilde{h}_L$ and $\tilde{h}_C$ is omitted in the description of local normal waves of both polarizations. The reflection coefficient $R^{\perp}_{h_L}$ is independent of the total ionosphere height. It depends only on variable $\mu = 2kC/\alpha$ alone, so $R^{\perp}_{h_L}(0) = -1$.

[20] At $|\mu| \ll 1$

$$\ln(-R^{\perp}_{h_L}) \cong -i\mu^3\psi''(1)/3 =$$

$$2i\mu^3\zeta(3)/3 = 0.8013709i\mu^3 \tag{10}$$

where $\psi''(1)$ is the third derivative of the logarithm of Gamma function with the argument equal to unity, and $\zeta(3)$ is the Riemann Zeta function. At $|\mu| \gg 1$, Stirling asymptotics can be used for the Gamma function entering the general expression for the reflection coefficient. Under this condition the reflection region is in the vicinity of the complex turning point h_0 [*Kirillov et al.*, 1993]

$$1 - \varepsilon'(h_0) = C^2 \tag{11}$$

$$h_0 = h_L + 2\alpha^{-1}(C_e + \ln\mu/2)$$

The coefficient of wave reflection from the ionosphere referred to the Earth's surface is written as

$$R^{\perp}_i(0) = \exp(-\frac{i\pi}{2} + 2ikCh^{\perp}_{\mathrm{ph}}) \tag{12}$$

where

$$h^{\perp}_{\mathrm{ph}} = h_0 - 2\alpha^{-1}(1-\ln 2) = h_0 - 0.613706\alpha^{-1}$$

[21] To find the normalization integral, it is also necessary to calculate the derivative of the ionospheric reflection coefficient near the Earth's surface with respect to C

$$\frac{\partial}{\partial C}R^{\perp}_i = 2ikh^{\perp}_{\mathrm{tr}}R^{\perp}_i \tag{13}$$

where $h^{\perp}_{\mathrm{tr}} = h_L + [2C_e + \psi(1+i\mu) + \psi(1-i\mu)]$ is the triangulation height. At $|\mu| \ll 1$ $h^{\perp}_{\mathrm{tr}} = h_L + 1.202056\mu^2\alpha^{-1}$ and at $|\mu| \gg 1$ and $\mathrm{Re}\mu > 0$ $h^{\perp}_{\mathrm{tr}} \cong h_0 + 1.386294\alpha^{-1}$. The real part of the triangulation height $h^{\perp}_{\mathrm{tr}}$ is somewhat higher than the real part of the turning point height h_0, and the real part of the phase height $h^{\perp}_{\mathrm{ph}}$ is somewhat lower.

[22] We assume that the complex dielectric permeability of the ground is constant and is equal to $\varepsilon'_g = \varepsilon_g + i\sigma/(\omega\omega_0)$. We also assume that at depth h_g there is an ideally reflecting surface, the introduction of which allows us to avoid the appearance of a continuous part in the spectrum of normal waves [*Kirillov and Kopeykin*, 1989]. In addition, the ideally reflecting surface at depths 70–100 km simulates the Earth's mantle.

[23] The impedance of a horizontally polarized wave at the Earth's surface is given by

$$\delta^{\perp}_g = -\frac{i}{\sqrt{\varepsilon'_g - S^2}}\tan\left(kh_g\sqrt{\varepsilon'_g - S^2}\right) \tag{14}$$

The coefficient of reflection from the Earth's surface in the case the wave propagates downward is

$$-R^{\perp}_g = \frac{1 - C\delta^{\perp}_g}{1 + C\delta^{\perp}_g}$$

or

$$-R^{\perp}_g = \frac{R^{\infty}_{\perp} + \exp(2ikh_g)\sqrt{\varepsilon'_g - S^2}}{1 + R^{\infty}_{\perp}\exp(2ikh_g)\sqrt{\varepsilon'_g - S^2}}$$

where $R^{\infty}_{\perp} = (\varepsilon'_g - 1)(C + \sqrt{\varepsilon'_g - S^2})^{-2}$ is the wave reflection coefficient from an infinitely conducting surface.

[24] The characteristic equation for finding parameters of normal waves can be represented in the form $R^{\perp}_g R^{\perp}_i = 1$ or, after taking the logarithm, as

$$Ckh_L - \frac{i}{2}[\ln(-R^{\perp}_{h_L}) + \ln(-R^{\perp}_g)] = m\pi, \quad m = 1, 2, ... \tag{15}$$

Under conditions $|kh_g\sqrt{\varepsilon'_g - 1}| \gg 1$ and $|C^2| \ll |\varepsilon'_g - 1|$, the characteristic equation (15) has the approximate solution

$$C^{\perp}_m = m\pi k^{-1}\left(h_L + \frac{i}{k\sqrt{\varepsilon'_g - 1}}\right)^{-1} \tag{16}$$

from which one can see that the effective waveguide height is $\mathrm{Re}\,h_L + (2\omega\mu_0\sigma_g)^{-1/2}$. Thus, under these conditions, the effective waveguide height also depends on the Earth's skin layer. As the mode number increases, the reflection height region rises to the turning point region and C^2 becomes comparable to $|\varepsilon'_g - 1|$, which gives the eigenvalues differing from equation (16). The eigenvalues are found in this case by numerical solution of equation (15). As the mode number further increases, the inequality $|C^2_m| \gg |\varepsilon'_g - 1|$ is satisfied, which allows one to describe the eigenvalues by the approximation

$$C^{\perp}_m = \frac{(m - 1/4)\pi}{k(h^{\perp}_{\mathrm{ph}} + h_g)} \tag{17}$$

under the condition

$$\pi^2 h_g m|\mathrm{Re}h^{\perp}_{\mathrm{ph}} + h_g|^{-2} < 2\ln[m\pi k^{-1}(\mathrm{Re}h^{\perp}_{\mathrm{ph}} +$$

$$h_c|\sqrt{\varepsilon'_g - 1}|)^{-1}]$$

In this case the waveguide is formed between the height $\mathrm{Re}h^{\perp}_{\mathrm{ph}}$ and depth h_g.

[25] For horizontally polarized electromagnetic waves, the eigenfunctions in which the field is expanded have the meaning of the horizontal component E_y (9). Let us denote them as $U^{\perp}_m(z)$. It is convenient to introduce the normalization integral by the relation

$$h^{\perp}_{N,m} = \left[\frac{1}{C^{\perp}_m k}\frac{dU^{\perp}_m}{dz}(0)\right]^{-1}\int_{-h_g}^{\infty} U^{\perp 2}_m(z)dz =$$

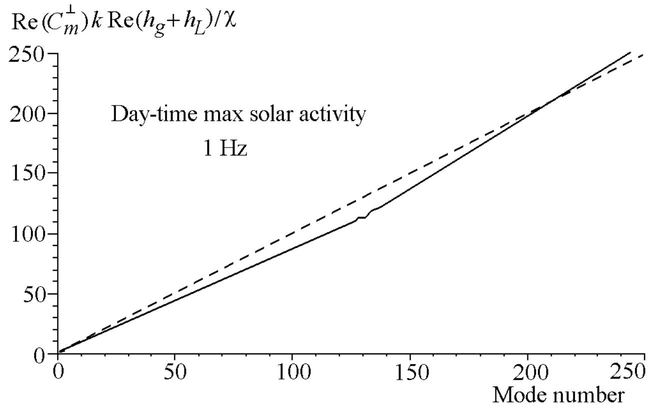

Figure 5. Real parts of eigenvalues of the first 250 numbers.

$$\left(\frac{C_m^\perp \delta_g^\perp (C_m^\perp)}{U_m^\perp(0)}\right)^2 \int_{-h_g}^{\infty} U_m^{\perp 2}(z)dz$$

The availability of analytical expressions for $R_i^\perp(C)$ and $R_g^\perp(C)$ allows one to find the analytical expression for the normalization integral

$$2h_{N,m}^\perp = \frac{\varepsilon_g' - 1}{\varepsilon_g' - (S_m^\perp)^2}\left(h_{\rm tr}^\perp + \frac{i\delta_g^\perp}{k}\right) +$$

$$\frac{(C_m^\perp)^2(h_{\rm tr}^\perp + h_g)}{(\varepsilon_g' - (S_m^\perp)^2)\cos^2\left(kh_g\sqrt{\varepsilon_g' - (S_m^\perp)^2}\right)}$$

The convenience of the normalization we have chosen is due to the fact that in the region where the eigenvalues are described by equation (16), that is, when the waveguide is formed by the Earth's surface and the height of the wave reflection from the ionosphere, the normalization integral $h_{N,m}^\perp$ is approximately equal to $2h_{N,m}^\perp \cong h_{\rm tr}^\perp + i\delta_g^\perp k^{-1}$. In the region for which formula (14) is valid, that is, when the waveguide is efficiently formed by the reflection height and the depth at which the ideally reflecting surface is assumed to be lying, the normalization integral is proportional, as before, to the waveguide thickness $h_{\rm tr}^\perp + h_g$.

$$2h_{N,m}^\perp \cong (h_{\rm tr}^\perp + h_g)\cos^{-2}\left(kh_g\sqrt{\varepsilon_g' - (S_m^\perp)^2}\right)$$

The presence of an additional multiplier is explained by the fact that the function to which the integral of the square of the eigenfunction is referred is not distinguished by the physics of waveguide formation when its width is equal to the sum of heights $h_{\rm tr}^\perp + h_g$.

[26] The graphs of eigenvalues and corresponding normalization integrals as functions of mode number are given in Figures 5–8. The eigenvalues and normalization integrals of the modes with a horizontal polarization are characterized by a complicated dependence on the mode number.

4. Reflection of Vertically Polarized Waves at $2k/\alpha < 0.1$ and Small Parameter $2kC/\alpha$

[27] We assume that $\partial/\partial x = ikS$ and $\partial/\partial y = 0$. Then for the vertically polarized electromagnetic field, the system of Maxwell equations yields the following system of ordinary differential equations with respect to the vertical z coordinate:

$$\frac{1}{ik}\frac{d}{dz}E_x = (1 - S^2/\varepsilon')Z_0 H_y$$

$$\frac{1}{ik}\frac{d}{dz}(Z_0 H_y) = \varepsilon' E_x \tag{18}$$

$$E_z = -\frac{S}{\varepsilon'}Z_0 H_y$$

$$Z_0 H_z = Z_0 H_x = E_y = 0$$

It is assumed that there is an exponential conductivity profile in the ionosphere, i.e., $d/dz\ln(\varepsilon' - 1) = \alpha$ is independent of z.

[28] The problem on reflection of vertically polarized waves for the case of a restricted parameter C ($C^2 + S^2 = 1$) was described by *Greifinger and Greifinger* [1978, 1979] and *Kirillov and Pronin* [1974]. The ionospheric impedance $\delta_i^\parallel$ at the Earth's surface at $2k/\alpha < 0.1$ and $kh_C < 0.1$ is given in this case by

$$\delta_i^\parallel(C^2) = -ik(h_L - h_C S^2) \tag{19}$$

[29] Let us consider that at the height h the impedance $\delta_h^\parallel$ and between this height and the surface of the Earth there is vacuum. The ionospheric impedance at the Earth's surface is calculated from the impedance $\delta_h^\parallel$ as

$$\delta_i^\parallel(C^2) = \frac{\delta_h^\parallel - iC\tan(kCh)}{1 - i\dfrac{\delta_h^\parallel}{C}\tan(kCh)} \tag{20}$$

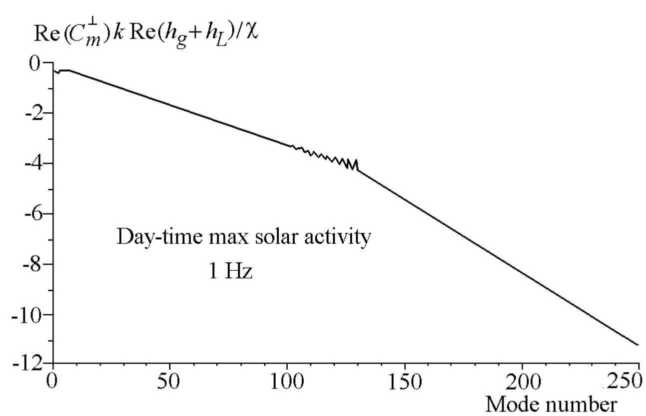

Figure 6. Imaginary parts of eigenvalues.

So, to describe the reflective properties of the ionosphere instead of impedance on the Earth's surface $\delta_i(C^2)$ it can be used the impedance $\delta_h^{\|}$ related to some height. Both descriptions are equivalent though the impedance at the Earth's surface and the corresponding impedance at the height h are different functions of C^2. If we refer the impedance to the complex height h_C, the impedance at this height will be given by

$$\delta_{h_C}^{\|} = -ik(h_L - h_C) \quad (21)$$

Under the condition $|kh_C| \ll 1$ according to equation (20) taking the impedance (21) we obtain on the Earth's surface the impedance (19). In the approximation used by *Greifinger and Greifinger* [1978, 1979] and *Kirillov and Pronin* [1974] the impedance, referred to the complex height h_C, at this height is independent of the wave incidence angle while the impedance at the Earth's surface depends on the angle of incidence. So, the problem can be formulated in such form as if inside the waveguide instead of the ionosphere there is vacuum and the complex height h_C is the upper boundary of the waveguide. Such method of introduction of the impedance at the complex height h_C allows us to discuss the wave reflection from the inhomogeneous ionosphere without restriction $|kh_C| \ll 1$. In the discussed approximation $2k/\alpha$ is a small parameter and $|C|$ is of order of unity.

[30] In the problem we plan to consider in the local modes $|C| \gg 1$. To expand the presented description on complex angles of incidence it is required to find the corrections to the impedance in the third order of smallness of $2k/\alpha$. With this purpose we divide the ionosphere into two parts: the lower region that corresponds to the interval $[0, h_1]$ and the upper region $z \geq h_1$. Let us choose the height h_1 in such a way that the modulus of $\xi = 2k\alpha^{-1}\sqrt{\varepsilon' - 1}$ should be small at his height and at the same time the following condition $|\xi_1| \gg 2k/\alpha$ should be fulfilled. The smallness of $|\xi|$ at the lower boundary of the upper region means that this boundary is situated below the reflection height of the wave. Discussing the upper region let us turn from the system of equations (18) to the equation for the impedance which can be presented in the form

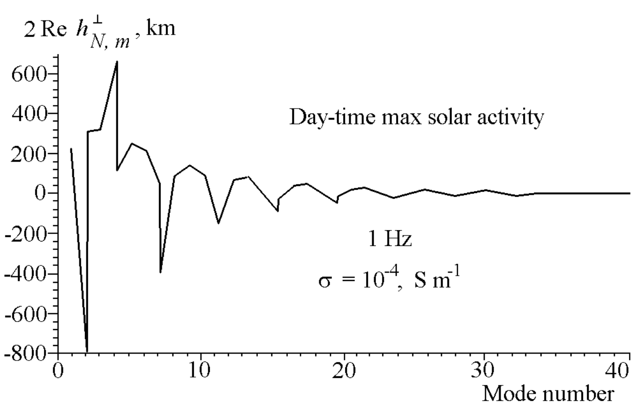

Figure 7. Real part of the normalization integral of the first 40 modes.

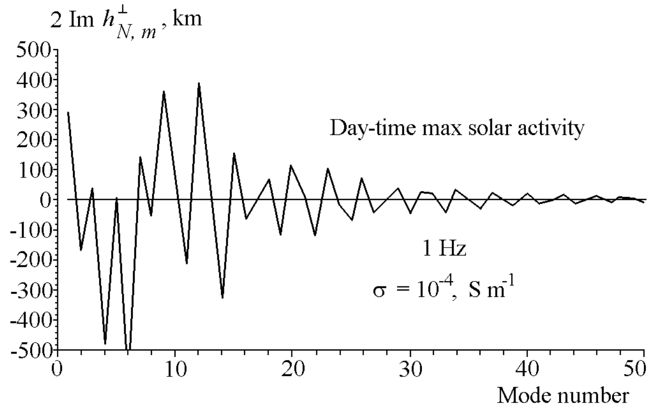

Figure 8. Imaginary part of the normalization integral.

$$-\frac{d}{d\alpha z}u = 1 - \frac{(2kS/\alpha)^2}{\xi^2 + 4k^2/\alpha^2} + \left(\frac{\xi^2}{4} + \frac{k^2}{\alpha^2}\right)u^2$$

$$\delta = \frac{E_x}{Z_0 H_y} = -\frac{ik}{\alpha}u \quad (22)$$

As compared with the system of differential equations (18), equation (22) is a nonlinear equation of the Rikkaty type.

[31] For normal incidence the distinction between the vertical and horizontal polarizations vanishes. The problem of reflection of the wave polarized horizontally in the case of exponential profile of conductivity is reduced to the Bessel equation as it follows from the solution of equation (18). For normal incidence the function u_0 is described by the formula

$$u_0 = -2\frac{H^{(1)}_{-2ik/\alpha}(\xi)}{\xi \dot{H}^{(1)}_{-2ik/\alpha}(\xi)} \quad (23)$$

The point above the symbol of the Hankel function means differentiation with respect to the argument of this function. In the case $|\xi| \ll 1$ it follows from formula (23) that

$$u_0 = X_1 - \frac{\xi^2}{4}(2 + 2X_1 + X_1^2) + \frac{k^2}{3\alpha^2}(-4\psi'' + X_1^3)$$

where ψ'' is the third derivative of the logarithm of the Gamma function with the argument equal to unity, $X_1 = \alpha(h_L - h_1)$. The last term in this formula presents the impedance at the normal incidence in the next order of smallness with respect to $2k/\alpha$. The derivative with respect to S^2 of the function u has the same order of smallness. Let us introduce $u_1 = (\alpha^2/k^2)(\partial u/\partial S^2)$. As a function of z, u_1 is the solution of the following linear differentiation equation:

$$\frac{d}{d\alpha z}u_1 = \frac{1}{\xi^2/4 + k^2/\alpha^2} - 2(\xi^2/4 + k^2/\alpha^2)uu_1$$

which is obtained from equation (22) by differentiation with respect to S^2. The solution of this equation that satisfies the limiting condition at $z \to +\infty$ is given by

$$u_1 = -\int_{\alpha z_1}^{\infty} \frac{d\alpha z}{\xi^2/4 + k^2/\alpha^2} \times$$

$$\exp\left[2\int_{\alpha z_1}^{\alpha z} (\xi'^2/4 + k^2/\alpha^2) u(\alpha z') d\alpha z'\right]$$

Within the whole region of integration the relation $2k/\alpha \ll |\xi|$ is fulfilled. This allows us to simplify the solution and to present it in the form

$$u_1 = -4\int_{\alpha z_1}^{\infty} \xi^{-2} d\alpha z \exp\left[0.5\int_{\alpha z_1}^{\alpha z} \xi'^2 u_0(\alpha z') d\alpha z'\right] \quad (24)$$

The derivative with respect to S^2 is considered at $S = 0$. It allows us to substitute u by u_0 determined by formula (23) in which it was taken $k = 0$. To determine the function u_1 at small αz it is possible to present the exponent under integral in (24) as a series and to retain the first two terms as the rest terms give infinitesimal input for $\alpha z \to 0$. As a result we arrive at the expression

$$u_1 = -4\xi_1^{-2} - 2\int_{\alpha z_1}^{\infty} u_0(\alpha z) d\alpha z$$

The investigation of asymptotic for $\alpha z \to 0$ allows us to obtain the following explicit expression for u_1

$$u_1 = -4\xi^{-2} + C - X_1^2$$

where the constant C is obtained as a limit

$$C = \lim_{\xi \to 0}\left(4\log^2(\gamma\xi/2) + 8\int_{\xi}^{\infty} \frac{K_0(\xi')}{\xi'^2 \dot{K}_0(\xi')} d\xi'\right)$$

and

$$X = \alpha h_L - \alpha h_1$$

$$K_0(\xi) = \frac{i\pi}{2} H_0^{(1)}(\xi \exp\frac{i\pi}{2})$$

The constant C was calculated numerically and its value was found to be -11.040912.

[32] Collecting together the obtained results we find for the impedance at the lower boundary of the upper region

$$\delta_1 = -\frac{ik}{\alpha}\left[X_1 - \left(\frac{\xi}{2}\right)^2 (2 + 2X_1 + X_1^2) + \right.$$

$$\left. \frac{k^2}{3\alpha^2}(-4\psi'' + X_1^3) - \left(\frac{2kS}{\alpha\xi_1}\right)^2 + \left(\frac{kS}{\alpha}\right)^2 (C - X_1^2)\right] \quad (25)$$

[33] In the region below a height of h_1, the wave propagates upward and then reflects back. The description of the wave propagation in this region is not reduced to known special functions. *Kirillov and Pronin* [1974] used the method of successive approximations to get a rough estimate of the ionospheric impedance (19) at the Earth's surface level.

[34] In order to describe the dependence of impedance on S^2 at the complex height h_C, it is necessary to derive the expression for it in the third order of smallness with respect to $(2k/\alpha)$. Let us write the system of equations (18) in the form of a formally vector equation

$$\frac{1}{ik}\frac{d\mathbf{U}}{dz} = A\mathbf{U} \quad (26)$$

$$\mathbf{U} = \{E_x, Z_0 H_y\}^T \qquad A_{11} = A_{22} = 0$$

$$A_{12} = 1 - \frac{S^2}{\varepsilon'} \qquad A_{21} = \varepsilon'$$

Vector $\mathbf{U}$ at the Earth's surface is calculated from vector $\mathbf{U}_{h_1}$ at height h_1 by left multiplication by the matrix $K_m(0, h_1)$ which satisfies matrix differential equation (26) with respect to the first argument and is reduced to a unit matrix if the arguments are equal, i.e.,

$$\frac{1}{ik}\frac{d}{dz}K_m(z, h_1) = AK_m(z, h_1)$$

where $K_m(h_1, h_1) = 1$. In the back integration from the Earth's surface to the complex height h_C, it is assumed that the right-hand side of equation (26) corresponds to the vacuum (sign v), $A_{v,11} = A_{v,22} = 0$, $A_{v,12} = C^2$, and $A_{v,21} = 1$. Let us denote the transformation matrix in this case as $K_v(z, 0)$. Then the vector at height h_C sought for will be written as $U_{h_C} = K_v(h_C, 0)K_m(0, h_1)U_{h_1}$.

[35] Matrixes $K_m(z, h_1)$ and $K_v(z, 0)$ are found by successive approximations, i.e.,

$$K_m(0, h_1) = 1 + K_m^{(1)}(0, h_1) + K_m^{(2)}(0, h_1) + K_m^{(3)}(0, h_1)$$

where

$$K_m^{(1)}(0, h_1) = -ik\int_0^{h_1} A(z)dz$$

$$K_m^{(2)}(0, h_1) = -k^2\int_0^{h_1} A(z)dz \int_z^{h_1} A(z')dz'$$

and

$$K_m^{(3)}(0, h_1) = ik^3\int_0^{h_1} A(z)dz \int_z^{h_1} A(z')dz' \int_{z'}^{h_1} A(z'')dz''$$

Similarly,

$$K_v^{(1)}(h_C, 0) = ik\int_0^{h_C} A_v dz = ikh_C A_v$$

$$K_v^{(2)}(h_C, 0) = -(1/2)k^2 h_C^2 A_v^2$$

and

$$K_v^{(3)}(h_C, 0) = -(i/6)k^3 h_C^3 A_v^3$$

If we denote the matrix of transformation from h_1 to h_C through the Earth's surface as $K(h_C, h_1) = K_v(h_C, 0)K_m(0, h_1)$ and present it in the form of a series of successive approximations, its terms will be

$$K^{(1)} = K_v^{(1)} + K_m^{(1)}$$

$$K^{(2)} = K_v^{(2)} + K_m^{(2)} + K_v^{(1)} K_m^{(1)}$$

$$K^{(3)} = K_v^{(3)} + K_m^{(3)} + K_v^{(1)} K_m^{(2)} + K_v^{(2)} K_m^{(1)}$$

Cumbersome and labor-consuming calculations yield fairly simple expressions for the elements of the matrix $K^{(1)}$

$$K_{11}^{(1)} = K_{22}^{(1)} = 0$$

$$K_{12}^{(1)} = -ik(h_1 - h_C) - ikS^2\alpha^{-1}(\varepsilon_1' - 1)^{-1}$$

and

$$K_{21}^{(1)} = -ik(h_1 - h_C) - ik(\varepsilon_1' - 1)\alpha^{-1}$$

These elements are of the first order of smallness with respect to parameter $2k/\alpha$ and are independent of the total ionosphere height. They depend on the height difference alone.

[36] The matrix $K^{(2)}$ is of the second order of smallness and its nonzero elements are only on the principal diagonal

$$K_{11}^{(2)} + K_{22}^{(2)} = K_{12}^{(1)} K_{21}^{(1)}$$

$$K_{12}^{(2)} = K_{21}^{(2)} = 0$$

and

$$K_{22}^{(2)} = -(1/2)k^2(h_1 - h_C)^2 - k^2(\varepsilon_1' - 1)\alpha^{-2} +$$

$$k^2 S^2 \alpha^{-2}\left[\pi^2/6 + \alpha(h_1 - h_C) - \alpha(h_1 - h_C)(\varepsilon_1' - 1)^{-1}\right]$$

The elements of this matrix are also independent of the total effective ionosphere height. To obtain the impedance δ_{h_C} in the third order of smallness with respect to $2k/\alpha$, it is sufficient to calculate from the matrix of the third order of smallness the element $K_{12}^{(3)}$

$$K_{12}^{(3)} = i\frac{k^3}{\alpha^3}\left\{(1/6)\alpha^3(h_1 - h_C)^3 - \right.$$

$$2(\varepsilon_1' - 1) + \alpha(h_1 - h_C)(\varepsilon_1' - 1) +$$

$$S^2\left[1 + \pi^2/3 + (2 + \pi^2/6)\alpha(h_1 - h_C)\right] -$$

$$\left. S^4\left[\pi^2/3 + (\pi^2/6)\ln 2 + (3/2)\zeta(3) - 2\beta\right]\right\}$$

where number

$$\beta = \sum_{m=0}^{\infty}(-1)^m \frac{\alpha_m}{m+2} = 0.2695882$$

$$\alpha_m = \sum_{n=0}^{m}(n+1)^{-2}$$

and the expression in the brackets following S^4 is numerically equal to 5.693958.

[37] The impedance δ_{h_C} is calculated as

$$\delta_{h_C} = \delta_1 + K_{12}^{(1)} - 2K_{22}^{(2)}\delta_1 - K_{21}^{(1)}\delta_1^2 - K_{12}^{(1)}K_{22}^{(2)} + K_{12}^{(3)}$$

The calculations yield

$$\delta_{h_C} = -i\tan(kh_L^{**} - kh_C) +$$

$$ik^3 S^2 \alpha^{-3}(16.33078 + 5.289858X^2 - 5.693958S^2) \quad (27)$$

where $\alpha h_L^{**} = \alpha h_L - (4/3)\psi'' k^2\alpha^{-2}$, $X = \alpha h_L - \alpha h_C = \pi i - 2\log(yk/\alpha)$. The impedance δ_1, like the elements of matrix K, depends on the height h_1 at which the field sewing is performed. In the formula obtained for the impedance at normal incidence δ_{h_C} such dependence is absent, as it should be. This indicates that the procedure of taking into account the medium below the reflection region is correct.

[38] Thus, to describe the reflection of vertically polarized electromagnetic waves at finite S^2, expression (27) for the impedance δ_{h_C} referred to the complex height h_C has been derived. The accuracy of the formula is of the third order of smallness with respect to parameter $2k/\alpha$. As a function of S^2, the impedance is a second-order polynomial. The coefficients of this polynomial are functions of $2k\alpha$ because $\alpha(h_L - h_C) = i\pi - 2\ln(k\gamma/\alpha)$. The boundary of the applicability of the formula for δ_{h_C} with increasing S^2 has not been defined; however, it is clear that it exists. To determine the eigenvalues of local modes, ultrahigh C^2 and S^2 are required. Let us investigate these cases in more detail.

5. Reflection of Vertically Polarized Waves at $2k/\alpha < 0.1$ and Large Parameter $2kC/\alpha$

[39] Let us pass from the systems of equations for the electromagnetic field (18) to the second-order differential equation with respect to $Z_0 H_y$

$$\left[\varepsilon' \frac{d}{dz}\frac{1}{\varepsilon'}\frac{d}{dz} + k^2(\varepsilon' - 1 + C^2)\right] Z_0 H_y = 0 \quad (28)$$

By substituting $Z_0 H_y = V\sqrt{\varepsilon'}$, we get for the new function the second-order differential equation without the first derivative

$$\left[\frac{d^2}{dz^2} + k^2(\varepsilon' - 1 + C^2) + \right.$$

$$\left.\frac{1}{2\varepsilon'}\frac{d^2}{dz^2}\varepsilon' - \frac{3}{4(\varepsilon')^2}\left(\frac{d\varepsilon'}{dz}\right)^2\right] V = 0$$

For the exponential profile,

$$\frac{1}{2\varepsilon'}\frac{d^2}{dz^2}\varepsilon' - \frac{3}{4(\varepsilon')^2}\left(\frac{d\varepsilon'}{dz}\right)^2 =$$

$$\frac{\alpha^2}{2}\frac{\varepsilon' - 1}{\varepsilon'} -$$

$$\frac{3}{4}\left(\frac{\varepsilon' - 1}{\varepsilon'}\right)^2\Bigg|_{|\varepsilon' - 1| \gg 1} = -\frac{\alpha^2}{4} + \frac{\alpha^2}{\varepsilon' - 1}$$

Under this condition, the equation for V has the form

$$\left[\xi^{-1}\frac{d}{d\xi}\xi\frac{d}{d\xi} + 1 + \frac{4k^2C^2}{\alpha^2}\xi^{-2} - \right.$$

$$\left.\xi^{-2} + \frac{k^2}{\alpha^2}\left(\frac{\xi}{2}\right)^{-4}\right] V = 0$$

$$\xi = \frac{2k}{\alpha}\sqrt{\varepsilon' - 1} \qquad (29)$$

Under condition $C \gg |\xi_1^{-2}|$ at the lower boundary of the upper region the last term in this equation can be omitted. As a result we arrive at the Bessel equation with the index $\nu = \sqrt{1 - \mu^2}$, $\mu = 2kC/\alpha$. Thus, in the upper region we have the solution

$$Z_0 H_y = A H_\nu^{(1)}(\xi)\sqrt{\varepsilon'}$$

$$E_x = \frac{A}{ik\varepsilon'}\frac{d}{dz}H_\nu^{(1)}\sqrt{\varepsilon'}$$

In the vicinity of the lower boundary of then upper region the modulus of the argument of the Hankel function is small. This makes it possible to use the corresponding asymptotic

$$Z_0 H_y \approx (\varepsilon' - 1)^{(1-\nu)/2} -$$

$$e^{-i\nu\pi}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{\alpha}\right)^{2\nu}(\varepsilon' - 1)^{(1+\nu)/2} \qquad (30)$$

The first term can be interpreted as the incident field, whereas the expression $(\varepsilon' - 1)^{1/2+\nu/2}$ can be interpreted as a reflected field with a certain reflection coefficient.

[40] Equation (28) for the magnetic component $Z_0 H_y$ in the case of the exponential conductivity profile in the ionosphere is written as

$$\left[\frac{d^2}{d\eta^2} + \frac{1}{\eta(1-\eta)}\frac{d}{d\eta} + \frac{\mu^2}{4\eta^2} - \frac{k^2}{\alpha^2\eta}\right] Z_0 H_y = 0 \qquad (31)$$

where $\eta = 1 - \varepsilon'$. Equation (31) has two regular singular points at $\eta = 0$ and $\eta = 1$ and one irregular point at $\eta = \infty$. Let us use this equation in the region at the upper boundary of which $|\eta| \ll \mu^2\alpha^2/k^2$. In this case, equation (31) is transformed into the Riemann equation with three regular singular points 0, 1, and ∞ [*Whitteker and Watson*, 1927]. Then by substituting the variable $Z_0 H_y$ by $\eta^{i\mu/2}V$ we reduce the Riemann equation to the hypergeometric equation

$$\eta(1-\eta)\frac{d^2}{d\eta^2}V + (1 + i\mu - i\mu\eta)\frac{dV}{d\eta} + \frac{i\mu}{2}V = 0$$

with parameters

$$a = -1/2 + i\mu/2 + \nu/2$$

$$b = -1/2 + i\mu/2 - \nu/2$$

$$c = 1 + i\mu \qquad \nu = \sqrt{1 - \mu^2}$$

The neighborhood of the Earth's surface corresponds to the neighborhood of the zero regular singular point of the equation. The lower boundary of the reflection region corresponds to the neighborhood of an infinite singular point. The problem is to extend the known solution of the equation in the neighborhood of the infinite singular point (30) to the neighborhood of the zero singular point.

[41] To derive the solutions of equation (31) simplified by omitting the last term for the neighborhood of the infinite singular point, we consider its partial solutions

$$U_\infty^{(1)}(\eta) = (-\eta)^{(1-\nu)/2}F(a, 1 + a - c; 1 + a - b; \eta^{-1})$$

$$U_\infty^{(2)}(\eta) = (-\eta)^{(1+\nu)/2}F(b, 1 + b - c; 1 + b - a; \eta^{-1})$$

where

$$F(a, b; c; \eta) = 1 + \frac{ab}{c}\eta + \frac{a(a+1)b(b+1)}{c(c+1)2!}\eta^2 + ...$$

is the hypergeometric function. At $\eta \to \infty$ functions $U_\infty^{(1)}(\eta)$ and $U_\infty^{(2)}(\eta)$ behave as

$$U_\infty^{(1)}(\eta) \approx (-\eta)^{(1-\nu)/2} = (\varepsilon' - 1)^{1/2-\nu}$$

and

$$U_\infty^{(2)}(\eta) \approx (-\eta)^{(1+\nu)/2} = (\varepsilon' - 1)^{1/2+\nu}$$

which corresponds to the behavior of $Z_0 H_y$ in the lower part of the reflection region. Therefore we write

$$Z_0 H_y = U_\infty^{(1)}(\eta) + R_\infty U_\infty^{(2)}(\eta) \qquad (32)$$

where

$$R_\infty = -\mathrm{e}^{-i\nu\pi}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{k}{\alpha}\right)^{2\nu}$$

is the wave reflection coefficient at the lower boundary of the reflection region, $\nu = \sqrt{1-\mu^2}$. To describe the field behavior in the neighborhood of the zero singular point, we consider other partial solutions of the equation

$$U_0^{(1)}(\eta) = (-\eta)^{i\mu/2}F(a,b;c;\eta)$$

$$U_0^{(2)}(\eta) = (-\eta)^{-i\mu/2}F(a+1-c,b+1-c;2-c;\eta)$$

At low η these partial solutions behave as

$$U_0^{(1)}(\eta) \approx (-\eta)^{i\mu/2} = e^{ikCz}(-\eta_0)^{i\mu/2}$$

and

$$U_0^{(2)}(\eta) \approx (-\eta)^{-i\mu/2} = e^{-ikCz}(-\eta_0)^{-i\mu/2}$$

where η_0 is the variable corresponding to the variable η at the Earth's surface. The partial solution $U_0^{(1)}(\eta)$ is associated with the incident field, the solution $U_0^{(2)}(\eta)$ is associated with the reflected field. Let us present the solution for Z_0H_y through partial solutions of the equation $U_0^{(1)}(\eta)$ and $U_0^{(2)}(\eta)$, $Z_0H_y(\eta) = AU_0^{(1)}(\eta) + BU_0^{(2)}(\eta)$. Then the ionospheric reflection coefficient at the Earth's surface level determined from the formula $Z_0H_y = D[\exp(ikCz) + R_i^{\|}\exp(-ikCz)]$ can be expressed through the ratio B/A. Then we have

$$R_i^{\|} = (-\eta_0)^{-i\mu}B/A = R_{h_C}^{\|}\exp(2ikCh_C) \qquad (33)$$

where $R_{h_C}^{\|} = B/A$ is the coefficient of wave reflection from the ionosphere referred to the complex capacitance height h_C at which $\varepsilon'(h_C) - 1 = 1$.

[42] The relation between partial solutions $U_\infty^{(1)}(\eta)$, $U_\infty^{(2)}(\eta)$, and $U_0^{(1)}(\eta)$, $U_0^{(2)}(\eta)$ is known [*Whitteker and Watson*, 1927]

$$U_\infty^{(1)}(\eta) = \frac{\Gamma(1+a-b)\Gamma(1-c)}{\Gamma(1+a-c)\Gamma(1-b)}U_0^{(1)}(\eta) +$$

$$\frac{\Gamma(1+a-b)\Gamma(c-1)}{\Gamma(c-b)\Gamma(a)}U_0^{(2)}(\eta)$$

$$U_\infty^{(2)}(\eta) = \frac{\Gamma(1-a+b)\Gamma(1-c)}{\Gamma(1+b-c)\Gamma(1-a)}U_0^{(1)}(\eta) +$$

$$\frac{\Gamma(1-a+b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(b)}U_0^{(2)}(\eta) \qquad (34)$$

By combining formulae (32) and (34), we obtain the reflection coefficient $R_{h_C}^{\|}$ as a function of μ and parameter $2k/\alpha$

$$R_{h_C}^{\|} = A(\mu)\frac{1 - B^+(\mu)(k/\alpha)^{2\nu}\exp(-i\pi\nu)}{1 - B^-(\mu)(k/\alpha)^{2\nu}\exp(-i\pi\nu)} \qquad (35)$$

Here

$$A(\mu) = \frac{\Gamma(1+i\mu)}{\Gamma(1-i\mu)}\frac{\Gamma^2(1/2+\nu/2-i\mu/2)}{\Gamma^2(1/2+\nu/2+i\mu/2)}$$

$$B^\pm(\mu) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\Gamma^2(1-\nu)\Gamma^2(1/2+\nu/2\pm i\mu/2)}{\Gamma^2(1+\nu)\Gamma^2(1/2-\nu/2\pm i\mu/2)}$$

At $|\mu| \ll 1$

$$A(\mu) \approx \exp[(i/4)(2\psi' - \psi'')\mu^3] = \exp(1.42349i\mu^3)$$

$$B^+ \approx B^- = -4\mu^{-4}$$

In expression (35) for the reflection coefficient $R_{h_C}^{\|}$, the terms with $B^\pm(\mu)$ are superior to unity in the vicinity of $\mu = 0$. In the vicinity of $\mu^2 = 2k/\alpha$ they are comparable to unity, and the reflection coefficient is close to the singular point. In the case $|\mu^2| < 2k/\alpha$, formula (35) is not applicable for description of reflection. If $|\mu^2| > 2k/\alpha$, and, simultaneously, $|\mu^2| \ll 1$, we get $\ln R_{h_C}^{\|} \cong i1.423490\mu^3$. This expression coincides with the expression for the logarithm of the reflection coefficient which can be obtained from the expression for the impedance (27) at small $2kC/\alpha$, but under the condition $|C| \gg 1$. In this case the dependence on the term with S^2 in the expression for the impedance disappears, and therefore there will be no uncertainty in the impedance and hence in the reflection coefficient.

[43] In order to find the upper boundary of the applicability of formula (35) with respect to parameter μ, calculations of the reflection coefficient from impedance (27) and formula (35) were carried out. The results are shown in Figures 9 and 10.

[44] In Figures 9 and 10 the values of parameter $2k/\alpha$ are given near the curves, the solid curves show the calculations from the impedance (27), and the dashed curves show the calculations using formula (35). It can be seen that the accuracy of calculations using the expression for low μ with the error within 10^{-2} is valid if $|\mu| < 0.3$. The approximate formula for high μ has the boundary of applicability from below, $|\mu| > 2\sqrt{2k/\alpha}$. If parameter $2k/\alpha$ takes the value $2k/\alpha = 0.1$, these expressions for the reflection coefficient overlap only with the accuracy of the order of 10^{-1}. The real part of the reflection coefficient characterizes its modulus. If μ is changed within unity, the modulus of the reflection coefficient achieves its maximum at some value of μ which depends on $2k/\alpha$. This value in the maximum is such one, that taking into account that the height h_C has negative imaginary part, the modulus of the reflection coefficient from the ionosphere at the Earth's surface appears to be larger than unity for real $C \gg 1$.

[45] The singular point for the reflection coefficient is the value of μ equal to unity. In this case

$$A(1) = \frac{\Gamma(1+i)}{\Gamma(1-i)}\frac{\Gamma^2(1/2-i/2)}{\Gamma^2(1/2+i/2)}$$

$$B^+(1) = B^-(1) = 1$$

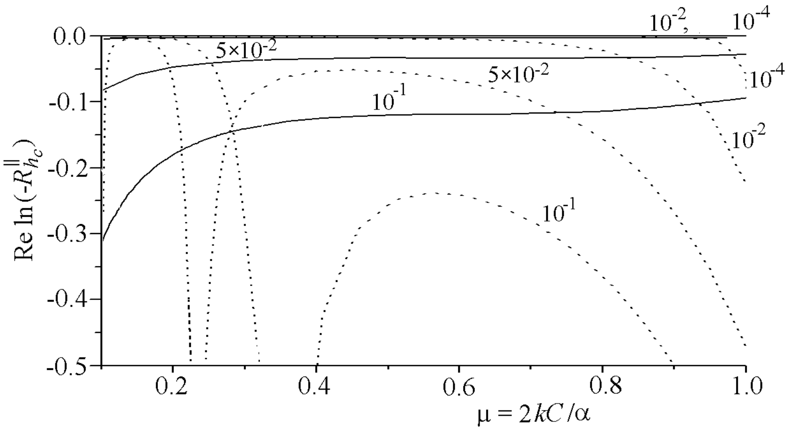

Figure 9. Comparison of $\mathrm{Re}\ln(R_{h_C}^{\parallel})$ calculated by using two equations for different μ and parameters $2k/\alpha$.

and the reflection coefficient can be calculated only after resolution of the uncertainty of the type 0/0

$$R_{h_C}^{\parallel}(1) = \frac{\alpha(h_L - h_C) - 2 - 2C_e - 2\psi(1/2 + i/2)}{\alpha(h_L - h_C) - 2 - 2C_e - 2\psi(1/2 - i/2)} A(1)$$

It is also possible to obtain a simpler expression by performing further transformations and calculations

$$R_{h_C}^{\parallel}(1) = \frac{-2.57266 + i0.26027 + 2\ln\alpha/k}{-2.57266 + i6.02291 + 2\ln\alpha/k} \times$$

$$\exp(i2.39963)$$

At $|\mu| \gg 1$, the Stirling formula can be used for the Gamma functions entering the expression for the reflection coefficient (35), which will yield for coefficients A and $B^{\pm}$

$$\ln A \cong i\pi/2 + \ln 2 - \pi\mu$$

$$\ln B^{+} \cong -\ln 2 + 2i\mu\ln\mu - 2i\mu + \pi\mu$$

$$\ln B^{-} \cong \ln 2 + 2i\mu\ln\mu - 2i\mu - \pi\mu$$

At real positive μ, the modulus of coefficient A exponentially decreases, the modulus of B^{+} grows, and the modulus of B^{-} decreases with increasing μ. In this case the denominator in formula (35) can be replaced by unity, and for $R_{h_C}^{\parallel}$ we have approximately

$$R_{h_C}^{\parallel} \cong -i\left\{2 - \exp\left[2i\mu(-1 + \ln\alpha\mu/k)\right]\right\}\exp(-\pi\mu) \qquad (36)$$

It can be seen that at real and positive μ the reflection coefficient, by oscillating, exponentially decreases with increasing of μ. Such oscillations are usually observed when two levels of reflection are present. For the eigenvalues of vertically polarized local modes of sufficiently large numbers, the variable μ has a large positive real part and a small negative imaginary part. Under these conditions, the exponent in formula (36) is superior to two, and the reflection coefficient coincides with the asymptotics of the reflection coefficient

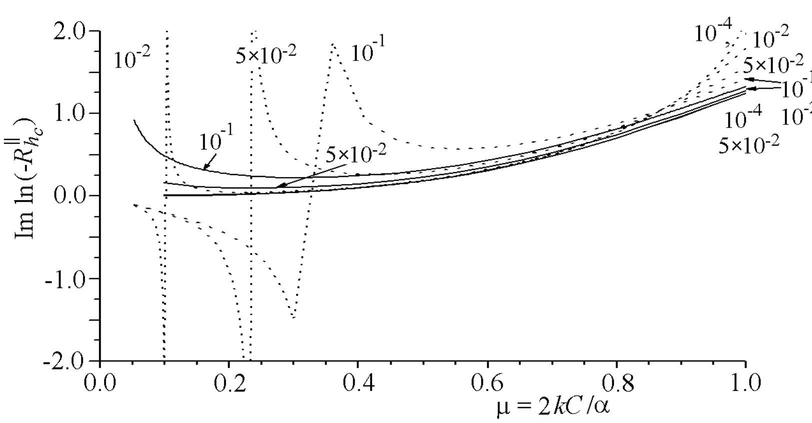

Figure 10. Comparison of $\mathrm{Im}\ln(R_{h_C}^{\parallel})$ calculated by using two equations for different μ and parameters $2k/\alpha$.

for the horizontally polarized waves (12), i.e.,

$$R_i^{\|} \approx R_i^{\perp}$$

at $\mathrm{Re}\mu \gg 1$ and $\mathrm{Im}\mu < 0$.

6. Vertically Polarized Modes

[46] The characteristic equation for the vertical polarization has the form

$$R_i^{\|}(C) R_g^{\|}(C) = 1 \tag{37}$$

where $R_i^{\|}(C)$ is the ionospheric reflection coefficient of the field at the Earth's surface level,

$$R_g^{\|} = \frac{C - \delta_g^{\|}}{C + \delta_g^{\|}}$$

is the coefficient of reflection from the Earth, and $\delta_g^{\|} = i\varepsilon_g'^{-1}\sqrt{\varepsilon_g' - S^2}\tan(kh_g\sqrt{\varepsilon_g' - S^2})$ is the ground impedance.

[47] For vertically polarized local modes, it can be approximately written $R_g^{\|} \approx 1 - 2\delta_g^{\|}/C$, $\delta_g^{\|} \approx \varepsilon_g'^{-1}\sqrt{\varepsilon_g' - S^2}$. By passing from reflection coefficients to logarithms, we get for the characteristic equation

$$kh_C C - (i/2)\ln R_{h_C}^{\|} + i\delta_g^{\|}/C = m\pi \tag{38}$$

[48] Eigenvalues of local modes of low numbers are in the region ($0 < \mathrm{Re}\mu < 1$). Since the imaginary part of the capacitance height is negative, the imaginary parts of the eigenvalues C_m of the first numbers can be positive.

[49] As the mode number increases, the eigenvalues fall into the region $\mathrm{Re}\mu \gg 1$, which allows one to replace the ionospheric reflection coefficient for the vertical polarization by the reflection coefficient for the horizontal polarization in characteristic equation (38). The characteristic equation in this case will acquire the form $kCh_{\mathrm{ph}}(C) + i\delta_g^{\|}/C = m\pi$, and the imaginary part of the eigenvalues will be negative.

[50] The eigenfunction of the vertically polarized modes coincides with the horizontal component of the magnetic field $Z_0 H_y$. It suffices to calculate the normalization integral in the limits of the ionospheric part of the waveguide

$$h_N^{\|} \equiv \frac{1}{H_y^2(0)} \int_0^{\infty} \varepsilon'^{-1} H_y(z) dz \Big|_{C=C_m} = \frac{i}{k}\frac{\partial}{\partial C^2}\delta_i^{\|} \tag{39}$$

The known analytical representations of the ionospheric reflection coefficient allow one to bring the normalization integral to the analytical form as well

$$2h_N^{\|} = h_{\mathrm{tr}}^{\|}[1 - (\delta_g^{\|}/C_m^{\|})^2 - i\delta_g^{\|}/(kC_m^{\|2})$$

where $ikh_{\mathrm{tr}}^{\|} \equiv \partial/\partial C \ln R_i^{\|}$. Approximately,

$$2h_N^{\|} \cong h_{\mathrm{tr}}^{\|}$$

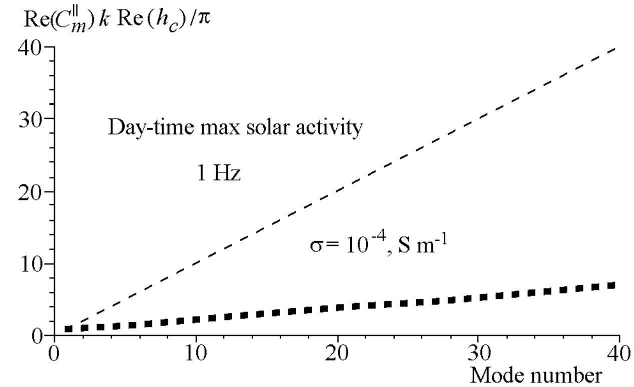

Figure 11. Real parts of eigenvalues for the first 40 vertically polarized modes.

which means that for vertically polarized modes the effective waveguide width coincides with the triangulation height. Below dependences of eigenvalues and triangulation height on the mode number are given (Figures 11–13).

[51] It can be seen that the effective height for the first modes approximately coincides with $\mathrm{Re}h_C$. The imaginary parts of the eigenvalues are negative. As the mode number increases, the effective height grows, it falls into the region of height $\mathrm{Re}h_L$ and then into the region of the heights of the turning point. For such large numbers, $\mathrm{Im}C_m$ are negative and grow in modulus with increasing mode number.

7. Representation of the Electromagnetic Field of a Horizontal Electric Dipole as Expansion in Normal Waves

[52] The field of the zero mode in the anisotropic approximation was given in section 1 (see equation (4)). The procedure of obtaining the representation of the electromagnetic field of a horizontal electric dipole by normal waves is rather cumbersome. It can be obtained by using the procedure

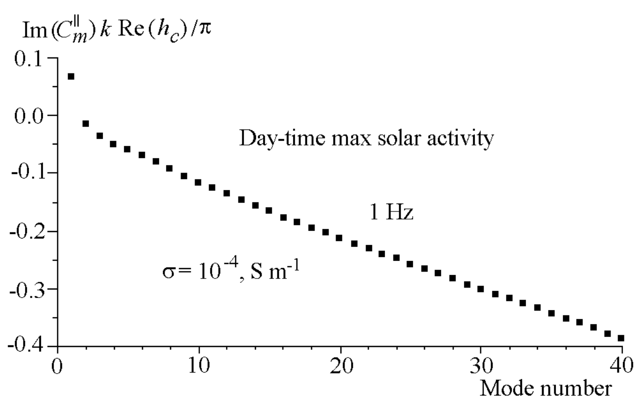

Figure 12. Imaginary parts of eigenvalues for the first 40 vertically polarized modes.

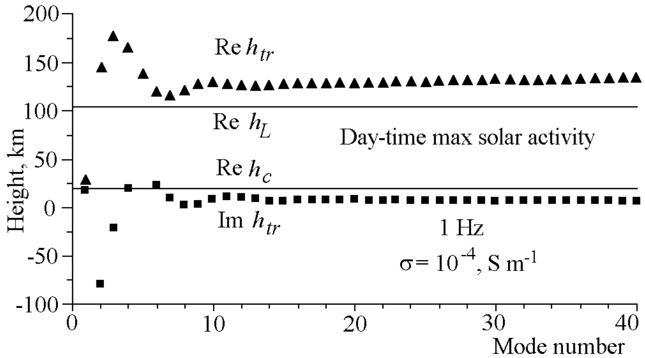

Figure 13. Doubled normalization integral for 40 vertically polarized modes.

given by *Kirillov* [1993] or independently, by considering the problem in the cylindrical coordinate system $\{\rho, \varphi, z\}$ related to the source. As a result, the field at the Earth's surface will be given by

$$\overline{E} = \overline{E}_0 + \overline{E}^{\|} + \overline{E}^{\perp}$$

$$\overline{H} = \overline{H}_0 + \overline{H}^{\|} + \overline{H}^{\perp} \tag{40}$$

$$E_z^{\|} = -\frac{i}{4} kIlZ_0 \delta_g \cos\varphi \sum_{m=1}^{\infty} (h_{N,m}^{\|})^{-1} S_m^{\|} \dot{H}_0^{(1)}(k\rho S_m^{\|})$$

$$E_\rho^{\|} = -Z_0 H_\varphi^{\|} \delta_g \qquad E_\varphi^{\|} = Z_0 H_\rho^{\|} \delta_g \tag{41}$$

$$H_z^{\|} = 0$$

$$H_\rho^{\|} = \frac{1}{4\rho} Il\delta_g \sin\varphi \sum_{m=1}^{\infty} (h_{N,m}^{\|})^{-1} (S_m^{\|})^{-1} \dot{H}_0^{(1)}(k\rho S_m^{\|})$$

$$H_\varphi^{\|} = \frac{1}{4} kIl\delta_g \cos\varphi \sum_{m=1}^{\infty} (h_{N,m}^{\|})^{-1} \dot{H}_0^{(1)}(k\rho S_m^{\|})$$

$$E_z^{\|} = 0$$

$$E_\rho^{\perp} = -\frac{1}{4\rho} Il\cos\varphi \sum_{m=1}^{\infty} \frac{(\delta_{g,m}^{\perp} C_m^{\perp})^2}{h_{N,m}^{\perp} S_m^{\perp}} \dot{H}_0^{(1)}(k\rho S_m^{\perp})$$

$$E_\varphi^{\perp} = \frac{1}{4} kIlZ_0 \sin\varphi \sum_{m=1}^{\infty} \frac{(\delta_{g,m}^{\perp} C_m^{\perp})^2}{h_{N,m}^{\perp}} \dot{H}_0^{(1)}(k\rho S_m^{\perp})$$

$$H_z^{\perp} = \frac{i}{4} kIl\sin\varphi \sum_{m=1}^{\infty} \frac{\delta_{g,m}^{\perp} (C_m^{\perp})^2 S_m^{\perp}}{h_{N,m}^{\perp}} \dot{H}_0^{(1)}(k\rho S_m^{\perp})$$

$$H_\rho^{\perp} = \frac{1}{4} kIl\sin\varphi \sum_{m=1}^{\infty} \frac{\delta_{g,m}^{\perp} (C_m^{\perp})^2}{h_{N,m}^{\perp}} \dot{H}_0^{(1)}(k\rho S_m^{\perp})$$

$$H_\varphi^{\perp} = \frac{1}{4\rho} Il\cos\varphi \sum_{m=1}^{\infty} \frac{\delta_{g,m}^{\perp} (C_m^{\perp})^2}{h_{N,m}^{\perp} S_m^{\perp}} \dot{H}_0^{(1)}(k\rho S_m^{\perp}) \tag{42}$$

For the modes with vertical polarization the ground impedance is nearly independent of the mode number, which allows one to express $E_\rho^{\|}$ and $E_\varphi^{\|}$ directly through $H_\varphi^{\|}$ and $H_\rho^{\|}$. Normalization integrals for both polarizations are of the order of the effective waveguide height. Horizontally polarized modes are proportional to $(C_m^{\perp})^2$, which are large numbers. Because of this, horizontally polarized modes make a larger contribution into the near-source field than vertically polarized modes.

[53] From formulae (40), (41), and (42), the $H_\varphi^{\|}$ component of the magnetic field of a horizontal electric dipole with a current moment of 10^7 A m was calculated as a function of distance. The angle between the dipole and path was 0 degrees. The modes were calculated for a daytime Earth–ionosphere waveguide with the lower wall characterized by a homogeneous conductivity of 10^{-4} S m^{-1} to a depth of 70 km and an infinite conductivity below. The model of the daytime ionosphere was taken for the conditions of solar activity maximum. The magnetic field relative to 1 A m^{-1} was calculated for a frequency of 1 Hz. The fields obtained in this way are given in Figures 14 and 15 in comparison with the fields calculated by using other formulae.

[54] The designations used in the figures are as follows: B_{sl} is the field calculated by using formulae (5); B_{an} is the field calculated from formulae (8); ρ^{-2} is the field calculated from formulae (6); ρ^{-3} is the field calculated from equation (7); Σ-mod is the field calculated from the sum of all the modes including local ones, and 0-mode is the field of one zero mode.

[55] One can see in Figure 14 that at distances from the source less than 60 km, the near-source magnetic field calculated under the condition of the ionosphere absence (B_{sl}) obeys the law of inverse square of the distance. In spite of such a short distance from the source, this field coincides with the field calculated from the waveguide modes in the presence of the ionosphere with an accuracy of not poorer than 0.1 dB, which is the evidence of a good quality of the work that has been done because the field of each mode, as an element of the sum, is ionosphere-dependent, while their sum does not depend on the ionosphere. Calculations of the near-source magnetic field from B_{an} give an error higher than 20 dB because near the source this field obeys the law of inverse cube of the distance but the real field obeys the law of inverse square.

[56] Figure 15 shows that as the distance from the source further increases, the near-source magnetic field calculated

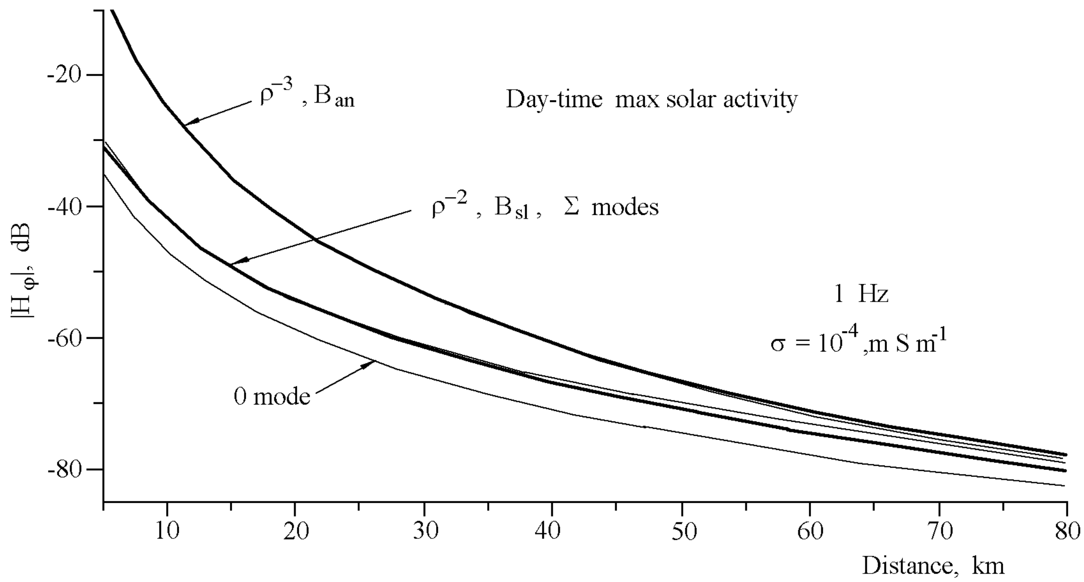

Figure 14. Horizontal magnetic field φ-component calculated by using different formulae in the range of distances from 5 to 80 km.

under the condition of the ionosphere absence (B_{sl}) departs from the law of inverse square of the distance and obeys the law of inverse cube of the distance beginning from a distance of the order of 75 km. Approximately at this distance, the influence of the ionosphere on the magnetic field begins to manifest itself, and the field calculated from the modes begins to deviate from the field derived from the model without the ionosphere (it increases). At a distance of 200 km, it transforms into the field of one normal wave. The magnetic field calculated from B_{an} does not pass exactly into the field of the anisotropic zero mode.

[57] The upper and lower boundaries of the transition region have been estimated with an accuracy of 0.2 dB in modulus. In the lower part of the transition region the field in the waveguide coincides with the specified accuracy with the field calculated from the model without the ionosphere, and in the upper part of the region the field coincides with the field of the leading normal wave. It was elucidated how positions of the transition region boundaries depend on frequency and the conductivity models of the Earth and ionosphere. The results are summarized in Table 2.

[58] It is evident from Table 2 that whatever the model

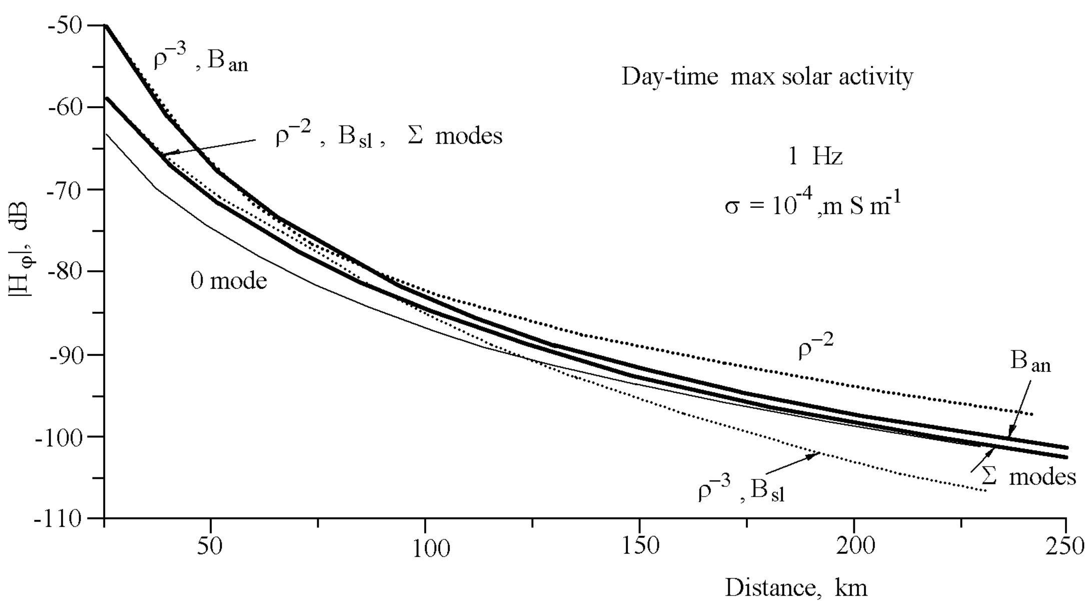

Figure 15. Horizontal magnetic field φ-component calculated by using different formulae in the range of distances from 20 to 250 km.

Table 2. Summary of Results [a]

F, Hz	σ_g, S m^{-1}	Re h_L, km	l_S, km	$\frac{\rho_1}{\mathrm{Re}\,h_L}$	$\frac{\rho_2}{\mathrm{Re}\,h_L}$	Σ/B_{ns}, dB	max(Σ/B_{ns}), dB	Conditions
200	10^{-4}	81.5	3.6	0.5	1.7	−0.3	0.9	day
100	10^{-4}	85.2	5.0	0.5	1.8	−0.2	−0.5	day
50	10^{-4}	87.3	7.1	0.5	1.8	−0.3	−0.5	day
10	10^{-4}	96.3	15.9	0.6	1.9	−0.3	−0.8	day
1	10^{-4}	104	50.3	1.0	2.3	−2.3	−2.3	day
0.2	10^{-4}	101	112	–	2.2	–	−4.5	day
0.2	10^{-3}	101	35.6	0.7	2.1	−1.2	−2.0	day
1	10^{-3}	104	15.9	0.6	1.9	−0.3	−0.7	day
10	10^{-3}	96.3	5.0	0.5	1.8	−0.2	−0.3	day
100	10^{-3}	85.2	1.6	0.5	1.8	−0.2	−0.4	day
100	10^{-3}	105	1.6	0.5	1.8	−0.2	−0.4	night
10	10^{-3}	83.6	5.0	0.7	1.8	−1.0	−1.4	night
1	10^{-3}	319	15.9	0.5	1.8	−0.2	−0.3	night
1	10^{-4}	319	50.3	0.6	1.9	−0.4	−0.8	night
10	10^{-4}	83.5	15.9	1.0	1.9	−1.6	−1.9	night
100	10^{-4}	105	5.0	0.5	1.8	−0.2	−0.5	night

[a] The designations are l_S is the thickness of the Earth's skin layer, $\rho_1/\mathrm{Re}\,h_L$ is the distance between the source and the lower boundary of the transition region in fractions of the inductance height, $\rho_2/\mathrm{Re}\,h_L$ is the distance between the source and the upper boundary of the transition region in fractions of the inductance height, Σ/B_{ns} is the ratio between the fields calculated from the sum of modes and from equation (8) in dB at the lower boundary of the transition region, max(Σ/B_{ns}) is the maximum value of this ratio in the transition region.

and frequency, the lower boundary of the transition region is approximately 0.5–0.7 of Re h_L. The upper boundary of the transition region was calculated from the distance beginning from which the contribution of local modes into the field becomes insignificant as compared with the contribution of the zero mode. It proved to be approximately equal to 1.8 Reh_L. Within the boundaries of the transition region, the discrepancy between the calculations from (B_{an}) and exact calculations does not exceed 2 dB.

[59] Comparison shows that the largest discrepancy occurs for the lower part of the transition region, where it reaches 8 dB. The deeper the lower boundary of the transition region goes into the zone adjacent to the source limited by the thickness of the Earth's skin layer, the larger the discrepancy.

8. Results

[60] Thus the near-Earth electromagnetic field of a horizontal electric dipole at frequencies 10^{-1} to 10^3 Hz from extremely short distances amounting to fractions of the Earth's skin layer thickness to the distances where the electromagnetic field nearly coincides with the field of the zero waveguide mode has been determined. The electromagnetic field is represented as expansion in waveguide modes of both polarizations in the entire range of distances mentioned above. The field of the zero mode is considered with due account of the anisotropy of the lower ionosphere. The field of local modes is treated in the isotropic approximation for the ionosphere described by the effective exponential conductivity profile. The obtained analytical description of the coefficient of reflection from the ionosphere at the Earth's surface level as a cosine of the wave incidence angle resulted in finding a huge number of local modes (of the order of several hundreds) which are needed to present the field in the near-source region.

[61] The transition region in which the field varies from the ionosphere-independent field to the field determined by the zero mode alone has been estimated. The description of the field in the transition region by modes has been compared with the description obtained by summing the fields of imaginary sources [*Bannister*, 1986].

[62] **Acknowledgment.** The authors thank E. D. Tereshchenko for stimulating this work.

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